A new model of dependent type theory

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Introduction

The goal of this short note is to present a new model of dependent type theory due to Christian Sattler. This is a obtained from (a variation of) the Dedekind model, that will be called the *Poset* model, by localising at a left exact modality. It is a model of univalence and higher inductive types, and seems to represent homotopy types in a satisfactory and constructive way.

1 The Poset model of type theory

We first present a variation of the Dedekind model of type theory.

Instead of working with *free* finite non degenerate distributive lattices like in the CCHM model, we work with *finitely presented* non degenerate distributive lattices.

By duality between finite posets and finite distributive lattices, this is equivalent to the presheaf model on the base category $\square$ of *finite non empty posets*.

We will write $Yo(X)$, or simply X for the presheaf represented by an object X.

This category contains as a full subcategory the category Δ of finite non empty *linear* posets, and the category Δ_+ with the same objects, where we restrict the maps to strictly monotone maps.

We write $[n]$ the linear poset with $n+1$ elements.

We consider the model based on the following cofibration classifier $\Phi(X)$, and interval $Yo([1])$. An element of $\Phi(X)$ is given by a finite collection Y_l of nonempty subsets of X (if the family is empty we get the empty sieve) and is the collection of maps of codomain X such that its image is contained in some Y_l.

We start by the presheaf model of type theory $P(\square)$, with base category $\square$. A context Γ is interpreted by a presheaf over $\square$ and a type over Γ by a presheaf on the category of elements of Γ. We write $\mathsf{Type}(\Gamma)$ the collection of types over Γ. Finally $\mathsf{Elem}(\Gamma, A)$ is the collection of global sections of the presheaf A.

If A in $\mathsf{Type}(\Gamma)$ and ρ in $\Gamma(X)$, we may write simply $A\rho$ for $A(X, \rho)$.

If A is in $\mathsf{Type}(\Gamma)$ a *composition operation* for A is an operation c_A which takes as argument ρ in $\Gamma(X \times [1])$ and Y_l nonempty subsets of X and u_0 in $A\rho(X \times 0)$ (the lower lid of the open box), and u_l in $A\rho(Y_l \times [1])$ (the cylinder of the box) and build an element $c_A(X, \rho, u_0, u_l)$ in $A\rho(X \times 1)$, element compatible with each u_l (the upper lid of the box). Furthermore this operation has to be *unifom*: we should have $c_A(X, \rho, u_0, u_l)f = c_A(Y, \rho(f \times [1]), u_0 f, u_l(f_l \times [1]))$ if $f : Y \to X$, with $f_l : f^{-1}(Y_l) \to Y_l$ induced by f. (There is then a similar operation swapping 0 and 1.)

We let $\mathsf{comp}(\Gamma, A)$ be the set of composition operations for A.

If X is linearly ordered with $|X| > 1$, any element x_0 in X defines a *horn*, which is the family of all proper subsets Y_l of X containing x_0. We say that A has *horn filling* if there is an operation $h_A(u_l)$ in $A(X, \rho)$ extending each compatible family of elements u_l in $A\rho(Y_l)$. *We don't require any uniformity condition.*

We let $\mathsf{horn}(\Gamma, A)$ be the set of horn filling operations for A.

We've shown in previous works that we get a model of dependent type theory with univalence, interpreting contexts Γ by presheaves on $\square$ and types on Γ as presheaves on the category of elements of Γ with a composition operation.

Lemma 1.1. *If A has a composition, it has horn filling. More precisely we have an operation* $\mathsf{comp}(\Gamma, A) \to \mathsf{horn}(\Gamma, A)$ *natural in* Γ.

Proof. Let X be linearly ordered with $|X| > 1$ and x_0 be an element of X. We assume given compatible elements $u(Y)$ in $A\rho(Y)$ for each subset Y of X containing x_0. If x_0 is not the top element of X_0 we can consider Z_0 the set of x in X such that $x \leqslant x_0$ and we have $u(Z_0)$ in $A\rho(Z_0)$. We have a map $h : X \times 0 \to Z_0$ by $h(x,0) = x$ if $x \leqslant x_0$ and $h(x,0) = x_0$ if $x_0 < x$. We build a a map $g(Y) : Y \times [1] \to Y$ by $g_l(x,0) = h(x,0)$ and $g_l(x,1) = x$. We define then $v(Y) = u(Y)g(Y)$ in $A\rho(Y)\mathsf{p}$ and $v(X \times 0) = u(Z_0)h$. By composition, we get an element $v(X \times 1)$ compatible with each $v(Y)$ which the horn filling of $u(Y)$.

There is a similar operation if x_0 is not the least element of X. □

We can think of $[n] \times [1]$ as a prism. We have $n+1$ subsets linearly ordered with $n+2$ elements. For instance, for $n = 2$, we have the subsets $(0,0),(1,0),(2,0),(2,1)$ and $(0,0),(1,0),(1,1),(2,1)$ and $(0,0),(0,1),(1,1),(2,1)$.

A difference between $P(\Delta)$ and $P(\square)$ is that in $P(\Delta)$ these subsets define a *covering* of $[n] \times [1]$ while they do not do it in $P(\square)$ for $n > 0$. (For $n = 1$, in $P(\square)$, a square is not the union of two triangles.)

2 The model over □ is not equivalent to spaces

We consider $\Theta = Yo(0,1)$. It can be checked that the diagonal map $\Theta \to \Theta^{\mathbf{I}}$ is an isomorphism. It follows that *any* dependent type over Θ has a composition structure.

We have two global points $1 \to \Theta$.

In particular we can consider P in $\mathsf{Type}(\Theta)$ such that $P\rho$ is the singleton $\{0\}$ if $\rho : X \to \{0,1\}$ is constant, and is empty otherwise. This defines a family of *strict* propositions.

For any global point $\sigma : 1 \to P$, we have that $P\sigma$ is the unit type.

However $\mathsf{Elem}(\Theta, P)$ is empty since for $\mathsf{id} : \{0,1\} \to \{0,1\}$ we have that $P\mathsf{id}$ is empty.

This shows that the associated model structure on $P(\square)$ cannot be equivalent to the one on spaces.

3 Lex operation

We've defined in [CRS] the notion of *lex operation*. It can be defined from a pointed pseudomorphism of a given model. For the present model, this is the monad induced from the inclusion functor $\Delta_+ \to \square$. Concretely, if F is a presheaf on $\square$, we define $DF(X)$ to be the set of functions $u(f)$, for $f : [n] \to X$, such that $u(f)g = u(fg)$ if $g : [m] \to [n]$ is strictly monotone.

This induces a lex operation [CRS] on types, by defining DA in $\mathsf{Type}(\Gamma)$ for A in $\mathsf{Type}(\Gamma)$ in the following way. An element u of $DA\rho$ is a function $u(f)$ in $A([n], \rho f)$ such that $u(f)g = u(fg)$ if $g : [m] \to [n]$ strictly monotone.

Lemma 3.1. *If A has a horn filling operation then DA has a composition operation. More precisely, we have an operation* $\mathsf{horn}(\Gamma, A) \to \mathsf{comp}(\Gamma, DA)$ *natural in* Γ.

Proof. Let ρ be an element in $\Gamma(X \times [1])$. Let u_0 be an element in $DA\rho(X \times 0)$, Y_l subsets of X and u_l an element in $DA\rho(Y_l \times [1])$ all compatible. (This defines an open box with a bottom lid.) We have to build u_1 in $DA\rho(X \times 1)$ compatible with all u_l. (This is the upper lid of the box.)

For this we define $u_1(f)$ for $f : [n] \to X$ by induction on n.

We strengthen the induction hypothesis by defining $u(P)$ in $A\rho(f \times [1])(P)$ for each linear nonempty subsets $P \subseteq [n] \times [1]$ in a compatible way.

We do the example $n = 2$.

If the image of f is a subset of some Y_l there is no choice and u_l gives $u(P)$.

Otherwise, we proceed as follows.

We first define $u((0,0),(1,0),(2,0),(2,1))$. By induction we have $u((0,0),(2,0),(2,1))$ and $u((1,0),(2,0),(2,1))$ and we are given $u((0,0),(1,0),(2,0))$, which is $u_0(f)$. By Horn filling, we have $u((0,0),(1,0),(2,0),(2,1))$.

Next we define $u((0,0),(1,0),(1,1),(2,1))$. We have by the first step $u((0,0),(1,0),(2,1))$ and by induction we have $u((0,0),(1,0),(1,1))$ and $u((1,0),(1,1),(2,1))$. Hence by Horn filling, we have $u((0,0),(1,0),(1,1),(2,1))$.

Finally, we define $u((0,0),(0,1),(1,1),(2,1))$. We have by the second step $u((0,0),(1,1),(2,1))$ and by induction we have $u((0,0),(0,1),(1,1))$ and $u((0,0),(0,1),(2,1))$. Hence by Horn filling, we have $u((0,0),(0,1),(1,1),(2,1))$.

In particular, we have $u((0,1),(1,1),(2,1))$ which is $u_1(f)$.

This composition operation is uniform in X. □

We have a map $\eta : A \to DA$ which is defined by $(\eta a)(f) = af$ if ρ in $\Gamma(X)$ and a in $A(X,\rho)$ and $f : [n] \to X$. We then have $(\eta a)(f)g = (\eta a)(fg)$ if $g : [m] \to [n]$ not necessarily strictly monotone.

If $\alpha : A \to B$ we define $D\alpha : DA \to DB$ by $(D\alpha u)(f) = \alpha u(f)$ for u in $DA(X,\rho)$.

4 Left exact modality

The work [CRS] gives a sufficient condition for a lex operation to be a left exact modality.

It is enough to have a path between η_{DA} and $D\eta_A$, of type $DA \to D^2A$, and that both maps are equivalences.

An element v in $D^2A(X,\rho)$ is a family $v(f)(g)$ in $A([m],\rho fg)$ indexed by maps $f : [n] \to X$ and $g : [m] \to [n]$, such that $v(f)(gh) = v(fg)(h)$ if $f : [n] \to X$ and $g : [m] \to [n]$ strictly motonotone and $h : [l] \to [m]$ and $v(f)(g)h = v(f)(gh)$ if $f : [n] \to X$ and $g : [m] \to [n]$ and $h : [l] \to [m]$ strictly motonotone.

We define a multiplication map $\mu : D^2A \to DA$ by $(\mu v)(f) = v(f)(\mathsf{id})$.

If $\alpha : A \to B$ we have $D\alpha \circ \mu = \mu \circ D^2\alpha : D^2A \to DB$ strictly. Indeed

$$((D\alpha \circ \mu)v)(f) = (D\alpha(\mu v))(f) = \alpha(\mu v)(f) = \alpha v(f)(\mathsf{id})$$

and

$$(\mu(D^2\alpha v))(f) = (D^2\alpha v)(f)(\mathsf{id}) = (D\alpha v(f))(\mathsf{id}) = \alpha v(f)(\mathsf{id})$$

We have $(\mu \circ \eta_{DA})u = (\mu \circ D\eta_A)u = u$ with a strict equality if u in $DA(X,\rho)$.

If I and J are linear order (maybe empty), we write $I + J$ the linear order where all elements of I are $<$ all elements of J.

Lemma 4.1. *There is a path between η_{DA} and $D\eta_A$. Furthermore this path can be built without using the composition for A.*

Proof. For u in $DA(X,\rho)$ we compute

$$(\eta_{DA}u)(f)(g) = uf(g) = u(fg)$$

and

$$(D\eta_A u)(f)(g) = (\eta_A u(f))(g) = u(f)g$$

We build an element v in $D^2A(X\times[1],\rho\mathsf{p})$ with $\mathsf{p} : X\times[1] \to X$, connecting $\eta_{DA}u$ and $D\eta_A u$. For this we consider $(f,h) : [n] \to X \times [1]$ and $g : [m] \to [n]$. We have $h : [n] \to [1]$, which gives a decomposition $[n] = A_0 + A_1$. This gives a corresponding decomposition $[m] = B_0 + B_1$ with maps $g_i : B_i \to A_i$. We can then define $v(f,h)(g) = u(f(g_0 + A_1))(B_0 + g_1)$. □

Corollary 4.1. *μ is the homotopy inverse of $D\eta_A$.*

Proof. We have seen that $\mu \circ D\eta_A = \mathsf{id}$ strictly.

In the other direction, we have $D\eta_A \circ \mu = \mu \circ D^2\eta_A$ strictly and by the previous Lemma, this is path equal to $\mu \circ D\eta_{DA} = \mathsf{id}$. □

It follows from this that $D\eta_A$ is an equivalence. Hence by [CRS], D defines a *left exact modality*.

We thus get [CRS] a new model of univalent type theory as the model of D-modal types.

Being a lex operation, it satisfies *strictness* conditions, for instance it is a strict monad. Using these strictness conditions, [CRS] explains how to interpret data types and higher inductive types in this model.

5 Some properties of the localised model

The model of D-modal types satisfies countable choice and Whitehead's principle.

One key property of this new model is the following.

Proposition 5.1. *Let A be an element of* $\mathsf{Type}(\Gamma)$ *which is a fibrant and is a proposition. If for any global point $\sigma : 1 \to \Gamma$ we have that $A\sigma$ is contractible then DA is contractible.*

We can use this model to define a Quillen model structure on the presheaf category $P(\Box)$. The cofibrations are the maps classified by Φ. The fibrations are the maps isomorphic to projections $\Gamma.A \to \Gamma$ where A is a (fibrant) type on Γ.

If C is a category, we can define $N(C)$ by $N(C)(X)$ set of functors $X \to C$. It should then be possible to prove Quillen theorem A for this model structure: if $F : C \to D$ is such that each comma category $d \downarrow F$ is contractible then $N(F)$ is an equivalence.

More generally, we expect that we can develop abstract homotopy theory constructively, using the language of type theory.

References

[CRS] Thierry Coquand, Fabian Ruch, and Christian Sattler. Constructive sheaf models of type theory. *Math. Struct. Comput. Sci.*, 31(9):979–1002, 2021.