The Univalence Principle

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Contents of this talk

What this talk is about

1. Quest for an equivalence-invariant foundation of mathematics
2. Overview: how can Voevodsky's Univalent Foundations (UF) be equivalence-invariant?
3. Our work on proving that UF are equivalence-invariant

What this talk is not about

- Precise definitions and proofs

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What this talk is not about

- Precise definitions and proofs

For precise mathematical results I refer to our book:

The Univalence Principle (arXiv:2102.06275)

Overview

Outline

Indiscernibility of identicals

Indiscernibility of identicals

$$x = y \quad \to \quad \forall P\, (P(x) \leftrightarrow P(y))$$

- Reasoning **in logic** is invariant under equality
- **In mathematics**, reasoning should be invariant under weaker notion of sameness!

Equivalence principle

Reasoning in mathematics should be **invariant under** the appropriate notion of **sameness**.

Invariance under sameness

Notion of sameness depends on the objects under consideration:

An equivalence principle for group theorists

$$G \cong H \to \forall \text{ group-theoretic properties } P, (P(G) \leftrightarrow P(H))$$

An equivalence principle for category theorists

$$A \simeq B \to \forall \text{ category-theoretic properties } P, (P(A) \leftrightarrow P(B))$$

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What are "structural" properties?

Violating the equivalence principle

What is **not** a structural property?

Exercise

Find a statement about categories that is not invariant under the equivalence of categories

Violating the equivalence principle

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A solution

"The category $\mathscr{C}$ has exactly one object."

Violating the equivalence principle

What is **not** a structural property?

Exercise

Find a statement about categories that is not invariant under the equivalence of categories

A solution

“The category $\mathscr{C}$ has exactly one object.”

Can we rule out such “non-structural” statements?

A language for invariant properties

Michael Makkai, *Towards a Categorical Foundation of Mathematics*:

> *The basic character of the Principle of Isomorphism is that of a* ***constraint on the language*** *of Abstract Mathematics; a welcome one, since it provides for the separation of sense from nonsense.*

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Makkai's FOLDS (First Order Logic with Dependent Sorts)

A language for categorical structures in which only invariant properties can be expressed

- FOLDS is not a foundation of mathematics
- Invariance only for properties, not for constructions

Univalent Foundations and the Univalence Principle

Voevodsky's goals

- Univalent Foundations as an "invariant language"
- Invariance not only for statements, but also for constructions: **any construction on objects in UF can be transported along equivalences of objects**

Essential ingredients for Univalent Foundations

- Martin-Löf identity type
- Voevodsky's univalence axiom

Univalent Foundations and the Univalence Principle

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In the rest of this talk

How is reasoning in Univalent Foundations invariant under equivalence?

Outline

Overview of types in type theory

Type former	Notation	(special case)	canonical term
Inhabitant	$a : A$		
Dependent type	$x : A \;\vdash\; B(x)$		
Sigma type	$\sum_{x:A} B(x)$	$A \times B$	(a, b)
Product type	$\prod_{x:A} B(x)$	$A \to B$	$\lambda(x : A).b$
Coproduct type	$A + B$		$\mathsf{inl}(a)$, $\mathsf{inr}(b)$
Identity type	$a = b$		$\mathsf{refl}(a) : a = a$
Universe	$\mathcal{U}$		
Base types	Nat, Bool, $\mathsf{1}$, $\mathsf{0}$		

Identity vs equality

Inhabitants of $x = y$ behave like equality in many ways

- $\mathsf{refl}(x) : x = x$
- $\mathsf{sym}(x, y) : x = y \to y = x$
- $\mathsf{trans}(x, y, z) : x = y \times y = z \to x = z$

Transport

$$\mathsf{transport} : x = y \to \prod_{B : A \to \mathcal{U}} \big(B(x) \simeq B(y)\big)$$

Inhabitants of $x = y$ behave **un**like equality

- Can iterate identity type
- Cannot show that any two identities are identical

The important features of univalent foundations

Homotopy levels

- Stratification of types according to “complexity” of their identity types
- Logic: notion of **propositions** given by one layer of this hierarchy

Univalence axiom

Specifies the identity type of a universe

Contractible types, propositions and sets

- A is **contractible**

$$\mathsf{isContr}(A) \;:\equiv\; \sum_{x:A}\prod_{y:A} y = x$$

- A is a **proposition**

$$\mathsf{isProp}(A) \;:\equiv\; \prod_{x,y:A} x = y$$

- A is a **set**

$$\mathsf{isSet}(A) \;:\equiv\; \prod_{x,y:A} \mathsf{isProp}(x = y)$$

$$\mathsf{Prop} \;:\equiv\; \sum_{X:\mathcal{U}} \mathsf{isProp}(X) \qquad \mathsf{Set} \;:\equiv\; \sum_{X:\mathcal{U}} \mathsf{isSet}(X)$$

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Equivalences

Definition

A map $f : A \to B$ is an **equivalence** if it has contractible fibers, i.e.,

$$\mathsf{isequiv}(f) \;:\equiv\; \prod_{b:B} \mathsf{isContr}\left(\sum_{a:A} f(a) = b\right)$$

The type of equivalences:

$$A \simeq B \;:\equiv\; \sum_{f:A\to B} \mathsf{isequiv}(f)$$

Outline

Different notions of equality

Synthetic vs. analytic equalities

In MLTT, we always have a **synthetic** equality type between $a, b : T$

$$a =_T b.$$

Depending on T, we might have a type of **analytic** equalities

$$a \simeq_T b.$$

Univalence Principle for T and $\simeq_T$ says that this map is an equivalence

$$(a =_T b) \to (a \simeq_T b)$$

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Univalence **Axiom**: for $T = \mathcal{U}$ and $(X \simeq_{\mathcal{U}} Y) =$ "equivalences $X \to Y$":

$$(X =_{\mathcal{U}} Y) \to (X \simeq_{\mathcal{U}} Y)$$

is an equivalence.

Transport along equivalence of types

$$x =_{\mathcal{U}} y \to \prod_{(P:\mathcal{U}\to\mathcal{U})} (P(x) \simeq P(y))$$

$$\mathsf{univalence}: \prod_{(x,y:\mathcal{U})} (x =_{\mathcal{U}} y \xrightarrow{\simeq} x \simeq y)$$

$$x \simeq y \to \prod_{(P:\mathcal{U}\to\mathcal{U})} (P(x) \simeq P(y))$$

Transport along biimplication

$$(P = Q) \to \prod_{(S:\mathsf{Prop}\to\mathcal{U})} (S(P) \simeq S(Q))$$

$$\mathsf{univalence} : \prod_{(P,Q:\mathsf{Prop})} ((P = Q) \xrightarrow{\simeq} (P \leftrightarrow Q))$$

$$(P \leftrightarrow Q) \to \prod_{(S:\mathsf{Prop}\to\mathcal{U})} (S(P) \simeq S(Q))$$

Transport along bijections

$$(X = Y) \to \prod_{(S:\mathsf{Set}\to\mathcal{U})} (P(X) \simeq P(Y))$$

$$\mathsf{univalence} : \prod_{(X,Y:\mathsf{Set})} ((X = Y) \xrightarrow{\simeq} (X \cong Y))$$

$$(X \cong Y) \to \prod_{(S:\mathsf{Set}\to\mathcal{U})} (P(X) \simeq P(Y))$$

Transport along isomorphism of groups

$$x =_{\mathsf{Grp}} y \to \prod_{(P:\mathsf{Grp}\to\mathcal{U})} (P(x) \simeq P(y))$$

$$\text{univalence} : \prod_{(x,y:\mathsf{Grp})} (x =_{\mathsf{Grp}} y \xrightarrow{\simeq} x \cong y)$$

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Structure Identity Principle

- One can show $(x =_{\mathsf{Grp}} y \xrightarrow{\simeq} x \cong y)$, using univalence for types
- Works similarly for many other structures **built from (types that are) sets**
- See Coquand & Danielsson and HoTT book (Section 9.9)

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What about things that form a **higher** category, e.g., categories themselves?

Categories in type theory

A **category** $\mathscr{C}$ is given by

- a type $\mathscr{C}_0 : \mathscr{U}$ of **objects**
- for any $a, b : \mathscr{C}_0$, a set $\mathscr{C}(a,b) : \mathscr{U}$ of **morphisms**
- operations: identity & composition

$$\begin{aligned} 1_a &: \mathscr{C}(a,a) \\ (\circ)_{a,b,c} &: \mathscr{C}(b,c) \to \mathscr{C}(a,b) \to \mathscr{C}(a,c) \end{aligned}$$

- axioms: unitality & associativity

$$1 \circ f = f \qquad f \circ 1 = f \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

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$$\mathrm{I}_a : \mathscr{C}(a,a)$$
$$(\circ)_{a,b,c} : \mathscr{C}(b,c) \to \mathscr{C}(a,b) \to \mathscr{C}(a,c)$$

- axioms: unitality & associativity

$$\mathrm{I} \circ f = f \qquad f \circ \mathrm{I} = f \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

A **univalent category** is a category $\mathscr{C}$ such that

$$(a = b) \to (a \cong b)$$

is an equivalence for all $a, b : \mathscr{C}_0$.

Local univalence implies global univalence

Theorem (A., Kapulkin, Shulman)

For categories A and B, let $A \simeq B$ denote the type of equivalences from A to B. If A and B are ***univalent****, we have*

$$(A =_{\mathsf{uCat}} B) \simeq (A \simeq B).$$

Transport along equivalence of **univalent** categories

$$x =_{\mathsf{uCat}} y \to \prod_{(P:\mathsf{uCat}\to\mathcal{U})} (P(x) \simeq P(y))$$

$$\text{univalence} : \prod_{(x,y:\mathsf{uCat})} (x =_{\mathsf{uCat}} y \xrightarrow{\simeq} x \simeq y)$$

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Univalence Principle for categories

- Holds only for categories that satisfy themselves a univalence condition: **local univalence implies global univalence**
- Univalent categories are the right notion of categories in Univalent Foundations

Our work: Univalence Principle

1. Define **signature**, **axiom**, and **theory** for mathematical structures, including higher-categorical ones
2. Given a theory $\mathscr{T} = (\mathscr{L}, T)$, define
   - $\mathscr{T}$-models
   - Univalence of $\mathscr{T}$-models
   - Equivalence between $\mathscr{T}$-models
3. Prove a univalence result for univalent $\mathscr{T}$-models:

$$\mathsf{univalence} : \prod_{(x,y:\mathsf{uMod}\mathscr{T})} ((x =_{\mathsf{uMod}\mathscr{T}} y) \xrightarrow{\sim} (x \simeq_{\mathsf{uMod}\mathscr{T}} y))$$

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Technical challenge

Define a notion of "isomorphism" (called **indiscernibility**) that

1. works for any signature/theory
2. specializes to categorical isomorphism for the theory of categories

1-categorical vs higer-categorical structures

When passing from set-level structures to higher-categorical structures, it **looks** like things get more complicated:

1. What is the role of the "local univalence" condition on categories?
2. Are higher-categorical structures fundamentally different from 1-categorical ones?

In the rest of the talk

1. Show what the local univalence condition means for 1-categorical structures
2. Our example: monoids

Outline

Monoids in type theory

In type theory, a monoid is a tuple $(M, \mu, e, \alpha, \lambda, \rho)$ where

1. $M : \mathbf{Set}$
2. $\mu : M \times M \to M$
3. $e : M$
4. $\alpha : \Pi_{(a,b,c:M)} \mu(\mu(a,b),c) = \mu(a,\mu(b,c))$
5. $\lambda : \Pi_{(a:M)} \mu(e,a) = a$
6. $\rho : \Pi_{(a:M)} \mu(a,e) = a$

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Why $M : \mathbf{Set}$?

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Why $M : \mathbf{Set}$?

Abstractly, a monoid is a (dependent) pair $(\mathit{data}, \mathit{proof})$ where

- *data* is 1.–3.
- *proof* is 4.–6.

The type of monoids

- We want two monoids $(\mathit{data}, \mathit{proof})$ and $(\mathit{data}', \mathit{proof}')$ to be the same if data is the same as data'.
- This is guaranteed when the types of proof and proof' are **propositions**.
- This in turn is guaranteed when M is a **set**.

Summarily:

$$\mathsf{Monoid} \;:\equiv\; \sum_{(M:\mathsf{Set})} \; \sum_{(\mu,e):\mathsf{MonoidStr}(M)} \mathsf{MonoidAxioms}(M, (\mu, e))$$

Can show

$$\mathsf{isProp}(\mathsf{MonoidAxioms}(M, (\mu, e)))$$

Monoid isomorphisms

Given $\mathbf{M} \equiv (M, \mu, e, \alpha, \lambda, \rho)$ and $\mathbf{M}' \equiv (M', \mu', e', \alpha', \lambda', \rho')$, a **monoid isomorphism** is a bijection $f : M \cong M'$ preserving μ and e.

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$$\begin{aligned}
\mathbf{M} = \mathbf{M}' &\simeq (M, \mu, e) = (M', \mu', e') \\
&\simeq \sum_{p:M=M'} (\mathsf{transport}^{Y \mapsto (Y \times Y \to Y)}(p, \mu) = \mu') \\
&\qquad\qquad \times (\mathsf{transport}^{Y \mapsto Y}(p, e) = e') \\
&\simeq \sum_{f:M \cong M'} \left(f \circ \mu \circ (f^{-1} \times f^{-1}) = \mu'\right) \\
&\qquad\qquad \times (f \circ e = e') \\
&\simeq \mathbf{M} \cong \mathbf{M}'
\end{aligned}$$

Transport along monoid isomorphism

We now have two ingredients:

1.
$$\mathbf{transport}_{\mathbf{M},\mathbf{M}'} : (\mathbf{M} = \mathbf{M}') \to \prod_{B:\mathbf{Monoid}\to\mathcal{U}} \big(B(\mathbf{M}) \simeq B(\mathbf{M}')\big)$$

2.
$$(\mathbf{M} = \mathbf{M}') \;\simeq\; (\mathbf{M} \cong \mathbf{M}')$$

Composing these, we get

$$\mathbf{transport}_{\mathsf{M},\mathsf{M}'} : (\mathsf{M} \cong \mathbf{M}') \to \prod_{B:\mathsf{Monoid}\to\mathcal{U}} \big(B(\mathbf{M}) \simeq B(\mathbf{M}')\big)$$

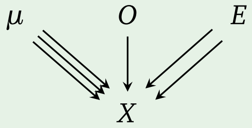
Outline

The signature of monoids

Example

Signature $\mathscr{L}_{\text{mon}}$ for a monoid:

$$\mathscr{L}_{\text{mon}} \quad =$$



A **structure** M for this signature consists of

1. a type MX
2. a family of types $M\mu(x, y, z)$ for $x, y, z : MX$
3. a family of types $MO(x)$ for $x : MX$
4. a family of types $ME(x, y)$ for $x, y : MX$

The theory of monoids

Not all structures represent monoids. **Axioms** specify those structures that are a monoid:

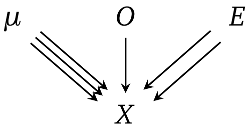
Axioms of a monoid

1. Monoid axioms:

$$\forall(x,y,z,z' : X).\mu(x,y,z) \to \mu(x,y,z') \to E(z,z')$$
$$\forall(x,y : X).\exists(z : X).\mu(x,y,z)$$
$$\forall(x,x',y,z : X).E(x,x') \to \mu(x,y,z) \to \mu(x',y,z)$$
$$\forall(x,y : X).E(x,y) \to E(y,x)$$
$$\ldots$$

2. "Homotopical axioms":
   - 2.1 MX is a set
   - 2.2 $M\mu(x,y,z)$, $MO(x)$, $ME(x,y)$ are pointwise propositions

Indiscernibility for elements of a monoid



Given $a, b : MX$, an **indiscernibility** $a \asymp b$ consists of "equivalences of types of everything above a and b"

$$M\mu(a, y, z) \simeq M\mu(b, y, z)$$
$$M\mu(x, a, z) \simeq M\mu(x, b, z)$$
$$M\mu(x, y, a) \simeq M\mu(x, y, b)$$
$$M\mu(a, a, z) \simeq M\mu(b, b, z)$$
$$\dots$$
$$MO(a) \simeq MO(b)$$
$$ME(a, y) \simeq ME(b, y)$$
$$\dots$$

Indiscernibility

1. $a \asymp b$ means that a and b behave in the same way within the structure.
2. In a model M of the theory of monoids, $a \asymp b$ reduces to $ME(a,b)$.
3. Definition of indiscernibility carries over to any $\mathscr{L}$, and any sort in $\mathscr{L}$.

Indiscernibility

1. $a \asymp b$ means that a and b behave in the same way within the structure.
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3. Definition of indiscernibility carries over to any $\mathscr{L}$, and any sort in $\mathscr{L}$.

Definition

1. Given $w, w' : M\mu(a,b,c)$, an indiscernibility $w \asymp w'$ is given by an equivalence

$$\mathbf{1} \simeq \mathbf{1}$$

(since there is nothing above μ in $\mathscr{L}_{\text{mon}}$). Hence $(w \asymp w') = \mathbf{1}$.

2. Similar for $w, w' : MO(a)$, and $w, w' : ME(a,b)$.

Univalence of models

Definition

A monoid M is **univalent** if the maps

$$(a = b) \to (a \asymp b) \tag{1}$$
$$(w = w') \to (w \asymp w') \tag{2}$$

are equivalences.

Univalence of models

Definition

A monoid M is **univalent** if the maps

$$(a = b) \to (a \asymp b) \tag{1}$$

$$(w = w') \to (w \asymp w') \tag{2}$$

are equivalences.

Observations

1. Since $(w \asymp w') = 1$, condition (2) is equivalent to $M\mu$, MO, and ME being pointwise **propositions**.
2. Since $(a \asymp b) = ME(a, b)$, condition (1) is equivalent to M being a **set with identity** $a = b$ **given by** $E(a, b)$.

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Summary: a univalent monoid in this sense is exactly the same a monoid as previously defined.

Equivalence of models

Given monoids M, N, an **equivalence** is

$$e_X : MX \to NX$$

$$e_\mu : \prod_{x,y,z:MX} M\mu(x,y,z) \to N\mu(ex,ey,ez)$$

$$e_O : \prod_{x:MX} MO(x) \to NO(ex)$$

$$e_E : \prod_{x,y:MX} : ME(x,y) \to NE(ex,ey)$$

such that e_X, $(e_\mu)_{x,y,z}$, $(e_O)_x$, and $(e_E)_{x,y}$ are (split-)surjective.

Observations

1. For univalent monoids, condition of e_E being split-surjective entails that e_X is injective.
2. An equivalence of univalent monoids is an isomorphism of sets that preserves multiplication and unit.

Univalence for univalent monoids

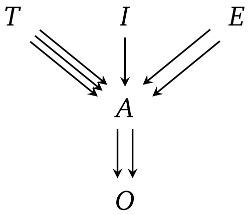
Theorem (Univalence Principle)

For univalent M and N,

$$(M = N) \simeq (M \simeq N)$$

Local univalence condition on categories

For the signature of categories,



an indiscernibility $a \asymp b$ in MO is the same as an isomorphism $a \cong b$, via a Yoneda-style argument.
The univalence condition at A says that A is a set with $f = g$ given by $ME(f, g)$.

Outline

Where does our work take place?

Working in **two-level type theory** of Annenkov, Capriotti, Kraus, Sattler.

- Univalent Foundations, embedded in an extensional type theory
- Universes $\mathscr{U} \hookrightarrow \mathscr{U}^s$
- $\mathscr{U}$ implements univalent type theory.
- Every type $T : \mathscr{U}^s$ is equipped with a strict equality type $a \equiv_T b$ with the usual rules for the identity type, but which also satisfies UIP.
- Signatures live in $\mathscr{U}^s$ (are meta-mathematical), but models and their morphisms live in $\mathscr{U}$ (are mathematical)

Signatures

- Signatures are abstract specification devices for mathematical structures
- We have two notions of signature.

“Diagram” Signatures

- Certain categories where
  - objects indicate sorts
  - morphisms indicate dependencies

+ Intuitive

– Complicated to reason about

“Functorial” Signatures

- (Co)inductively defined

+ Easy to reason about

– Difficult/unintuitive for specifying instances

Translation of Signatures

- We use diagram signatures for examples
- All results are proved for functorial signatures
- Algorithmic translation from diagram to functorial signatures
- Functorial signatures axiomatize the operation of “derivation”
- Functorial signatures are more general than diagram signatures

Diagram Signatures and Their Models

Definition (Diagram Signatures)

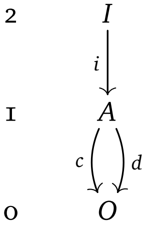
Too complicated, let's just draw some examples!

Definition (Models of a Diagram Signature)

Even more complicated. . .

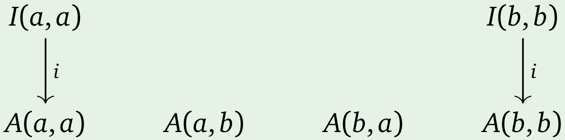
- Definition of model uses "derivation" of signatures
- Derivation does not preserve "finiteness" of signatures

Example of derivation



Example

In $\mathscr{L}_{\mathrm{rg}}$ we have $(\mathscr{L}_{\mathrm{rg}})_0 \equiv \{O\}$. Let M_0 be a (a function picking out) the two-element set $\{a, b\}$. Then $(\mathscr{L}_{\mathrm{rg}})'_{M_0}$ is the following signature, with four sorts of rank 0 and two sorts of rank 1:



Functorial Signatures

Observation: Essential Features of a Diagram Signature $\mathscr{L}$

- The type $\mathscr{L}(\mathrm{o})$ of non-dependent sorts
- The derived diagram signature $\mathscr{L}'_M$ for any $M : \mathscr{L}_{\mathrm{o}} \to \mathscr{U}$.

Definition (Functorial Signature, coinductively)

Consists of

- a type $\mathscr{L}_{\mathrm{o}}$
- for any $M : L_{\mathrm{o}} \to \mathscr{U}$, a functorial signature $\mathscr{L}'(M)$

- Definition can be made inductive by decorating it with a decreasing **height**.
- Need to define not just the pretype, but the strict **category** of functorial signatures (of height n).

Models of a Functorial Signature

Definition ($\mathscr{L}$-structure)

- Of $\mathscr{L}$ of height 0: a unique structure
- Of $\mathscr{L}$ of height $n+1$:
  1. A function $M : \mathscr{L}_0 \to \mathscr{U}$
  2. a structure of $\mathscr{L}'(M)$.

- Morphism of structures, composition, identity
- Pullback of structures along morphisms of signatures

Axioms and Theories

- An $\mathscr{L}$**-axiom** is a function $\mathbf{Struc}(\mathscr{L}) \to \mathsf{hProp}$.
- A **functorial theory** is a pair $(\mathscr{L}, T)$ of a functorial signature $\mathscr{L}$ and a family T of $\mathscr{L}$-axioms

Outline

Indiscernibility

Goal

Define a notion of "isomorphism" for elements of a structure

Definition (Indiscernibility)

Too complicated to write down, will be defined for structures of diagram signatures by example.

Definition (Univalence)

An $\mathscr{L}$-structure M is **univalent** if for any sort K of $\mathscr{L}$ and any $a, b : M(K)$, the map

$$a = b \to a \cong b$$

is an equivalence.

Results on Homotopy Levels

Theorem

If $\mathscr{L}$ has height $n+1$, $M : \mathbf{Struc}(\mathscr{L})$ is univalent, and $K : \mathscr{L}(0)$, then MK is an $(n-1)$-type.

Theorem

If $\mathscr{L}$ has height n, then the type of univalent $\mathscr{L}$-structures is an $(n-1)$-type.

Univalence Principle

Theorem

For any functorial signature $\mathscr{L}$ and $M, N : \mathbf{Struc}(\mathscr{L})$ that are both univalent, the canonical map

$$(M = N) \to (M \simeq N)$$

is an equivalence.

Other examples

- First-order logic (with equality)
- Higher-order logic, e.g., topological spaces, suplattices
- Categories
- Dagger categories
- (Ana)functors
- Profunctors
- Displayed categories / Fibrations
- Bicategories
- Double categories
- ...

Outline

Higher-Order Logic: Topological Spaces

- $M \mapsto (M \to \mathsf{hProp}) \to \mathsf{hProp}$
- Space M is such a structure by equipping it with the family of all supersets of the set of open subsets, i.e., a predicate that holds of X just when $U \in X$ for every open subset U of M.
- Morphism of structures: $f : M \to N$ such that if X contains all opens in M, then its image under f contains all opens in N, which is to say that $f^{-1}(U) \in X$ for all opens U in N. This is equivalent to saying that $f^{-1}(U)$ is open in M for all opens U in N, i.e., that f is continuous.

Suplattices

- $$\mathcal{L}'_M =_{\text{df}} (M \times M) + ((\textstyle\sum_{(A:\mathbf{Set})}(A \to M)) \times M)$$
- $M \times M$ stands for the partial ordering—(m, n) meaning $m \leq n$—whereas the second summand denotes suprema: (X, s) holds if and only if s is a supremum of the family X of elements of M.
- Structure M, then $m_1, m_2 : M$ are indiscernible if $m_1 \leq m_2$ and $m_2 \leq m_1$. (That m_1 and m_2 are suprema of exactly the same families X is then automatic.)
- Univalence at bottom level means that M is a set, and that the preorder $\leq$ on M is antisymmetric.
- Morphism of structures is sup-preserving morphism of preorders
- It is an equivalence if it is (split) surjective up to isomorphism and reflects the preorder (and hence also suprema of families).

Open Questions

- Completion operation for structures — turning a structure into a univalent one, universally? Ongoing work, e.g., by Kobe Wullaert.
- Univalence principle for structures of infinite height?
- Class of axioms that are invariant under weak equivalence?
- Signatures where functions are native, that is, not expressed as functional relations?
- Formalization? Ongoing work, e.g., by Elif Uskuplu.

References

- Coquand, Danielsson, “Isomorphism is equality”
- The HoTT book (Section 9.9 for Structure Identity Principle)
- Ahrens, Kapulkin, Shulman, “Univalent categories and the Rezk completion”
- Ahrens, North, Shulman, Tsementzis, “The Univalence Principle”