Homotopical Semantics of Type Theory

Steve Awodey

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Background

Connections have come to light between the **type theory** used in some proof systems (Coq, Agda, Lean, ...) and **homotopy theory**.

I. I will first sketch the **basic connection** between Martin-Löf's identity types and weak factorization systems.
II. Other type-theoretic ideas then lead to a Quillen model category, forming a **homotopical model of type theory**.
III. Finally, I will show how to extract a **strict model of type theory** from such a homotopical model.

Thierry's lectures will describe the **same constructions** in the language of type theory rather than category theory.

Martin-Löf Type Theory

Identity Types

Martin-Löf (1973) introduced the *identity type*, for terms $a, b : X$,

$$\mathsf{Id}_X(a, b)$$

Its rules preserved the constructive character of the system of type theory. But they also introduced some **intensionality**:

- terms $a, b : X$ identified by $p : \mathsf{Id}_X(a, b)$ may remain distinct,
- there may also be different $p, q : \mathsf{Id}_X(a, b)$,
- is there always a term $\alpha : \mathsf{Id}_{\mathsf{Id}_X(a,b)}(p, q)$?

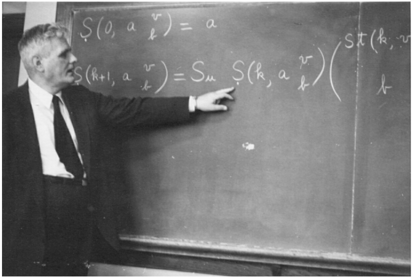
This system is used in computer proof systems like **Coq** because of its good computational properties, but its meaning was somewhat mysterious ...

The Topological Interpretation: Simple Types

Church showed that a numerical function is *definable* in the simply-typed λ-calculus just if it is *computable*.

Scott interpreted computability as **continuity**:

$$\begin{array}{rcl} \text{types} & \rightsquigarrow & \text{spaces} \\ \text{terms} & \rightsquigarrow & \text{continuous functions} \end{array}$$



The Homotopy Interpretation: Identity Types

We can **extend the topological interpretation** to identity types:

$$\begin{array}{rcl} \text{types } X & \rightsquigarrow & \text{spaces} \\ \text{terms } t : X \to Y & \rightsquigarrow & \text{continuous functions} \\ \text{identities } p : \mathrm{Id}_X(a, b) & \rightsquigarrow & \text{paths } p : a \sim b \end{array}$$

In topology, a **path** $p : a \sim b$ from point a to point b in a space X is a continuous function

$$p : [0, 1] \to X$$

with $p(0) = a$ and $p(1) = b$.

The Homotopy Interpretation: Identity Types

The identity types endow each type X with **higher structure**.

$$
\begin{aligned}
a, b &: X \\
p, q &: \mathrm{Id}_X(a, b) \\
\alpha, \beta &: \mathrm{Id}_{\mathrm{Id}_X(a,b)}(p, q) \\
&\dots
\end{aligned}
$$

The higher identity terms are interpreted as **homotopies**:

$$
\begin{array}{rcl}
X & \rightsquigarrow & \text{space} \\
a, b : X & \rightsquigarrow & \text{points of } X \\
p : \mathrm{Id}_X(a, b) & \rightsquigarrow & \text{paths } p : a \sim b \\
\alpha : \mathrm{Id}_{\mathrm{Id}_X(a,b)}(p, q) & \rightsquigarrow & \text{homotopies } \alpha : p \approx q \\
& \dots &
\end{array}
$$

The Homotopy Interpretation: Type Dependency

The interpretation of identity terms as *paths* requires dependent types to be interpreted as *fibrations*.

A family of types $x : X \vdash F(x)$ will be a **bundle of spaces**, i.e. a continuous map:

$$x : X \vdash F(x) \quad \rightsquigarrow \quad \begin{array}{c} F \\ \downarrow \\ X \end{array}$$

The Homotopy Interpretation: Type Dependency

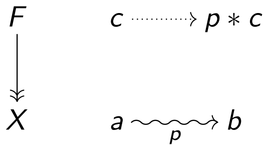
The rules for identity types permit the inference:

$$\frac{p : \mathrm{Id}_X(a,b) \qquad c : F(a)}{p * c : F(b)}$$

Logically, this just says the predicate $F(x)$ *respects identity*:

$$\mathrm{Id}_X(a,b) \ \& \ F(a) \ \Rightarrow \ F(b)$$

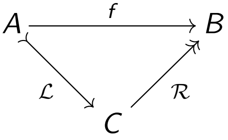
Topologically, it is the *path lifting property* of **fibrations**:



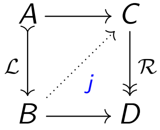
Definition: Weak Factorization System

A **weak factorization system** on a category consists of two classes of maps $(\mathcal{L}, \mathcal{R})$ such that:

- every map factors as an $\mathcal{L}$ followed by an $\mathcal{R}$,



- every commutative square with an $\mathcal{L}$ and an $\mathcal{R}$ thus,



has a **diagonal filler** j making it commute.

Basic HoTT: Identity Types

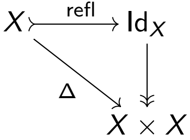
Theorem (A-Warren, 2006)

Martin-Löf's **rules for identity types** *interpret into any wfs.*

Proof.

- Types $\Gamma \vdash X$ are interpreted as $\mathcal{R}$-maps.
- Terms $\Gamma \vdash t : X$ are interpreted as sections.
- Factoring $\Delta : X \longrightarrow X \times X$ interprets Formation and Intro,

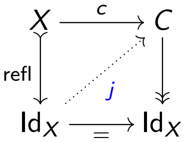
$$x, y : X \vdash \mathsf{Id}_X(x, y) \text{ type} \qquad\qquad x : X \vdash \mathsf{refl}(x) : \mathsf{Id}_X(x, x)$$



Basic HoTT: Identity Types

- Elimination assumes a commutative square of the form,

$$x : X \vdash c(x) : C(x, x, \mathsf{refl}(x))$$



The diagonal filler j is the Elim-term,

$$x, y : X, z : \mathsf{Id}_X(x, y) \vdash j(x, y, z; c) : C(x, y, z)\,.$$

- The upper triangle is the Computation rule,

$$c = j \circ \mathsf{refl}\,.$$

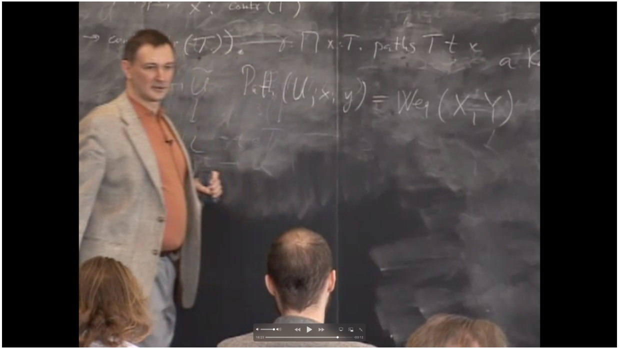
□

Univalent HoTT

Voevodsky added the *Univalence Axiom* to HoTT (in 2010)

$$\mathsf{Id}(X, Y) \simeq (X \simeq Y)$$

and constructed a model in simplicial sets using the **Quillen model structure**.



Definition: Quillen Model Structure

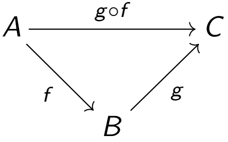
A **Quillen model structure** on a category $\mathcal{E}$ consists of *three* classes of maps

$$(\mathcal{C}, \mathcal{W}, \mathcal{F})$$

such that:

(*a*) $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems,

(*b*) $\mathcal{W}$ has the 2-of-3 property:



if any 2 of the above arrows are in it, then all 3 are.

Homotopical Models of Type Theory

Definition

A **model of HoTT** in a topos $\mathcal{E}$ is:

i) a **Quillen model structure** $(\mathcal{C}, \mathcal{W}, \mathcal{F})$:
   (a) $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems,
   (b) $\mathcal{W}$ has the 2-of-3 property,

ii) satisfying the **Frobenius** condition,

iii) with a **univalent universe** of fibrations $\dot{\mathrm{U}} \twoheadrightarrow \mathrm{U}$,

iv) such that U is a **fibrant object**.

Remark

We'll see that condition (i b) follows from the others.

Part II

Cubical sets

As the category $\mathcal{E}$ we take the **cubical sets**

$$\mathsf{cSet} = \mathsf{Set}^{\Box^{\mathrm{op}}} .$$

The **cube category** $\Box$ can be e.g. the dual of the finitely generated free distributive lattices,

$$\Box^{\mathsf{op}} := \mathsf{DLat}_{fgf} .$$

$\Box$ is closed under finite products and contains a bipointed object,

$$[0] \rightrightarrows [1] .$$

The interval $\mathbb{I}$

The interval $1 = \mathrm{y}[0] \rightrightarrows \mathrm{y}[1] = \mathbb{I}$ in cSet provides, for every X:

- a **cylinder**

$$X + X \rightarrowtail \mathbb{I} \times X \, .$$

- a **path object**

$$X^{\mathbb{I}} \twoheadrightarrow X \times X \, .$$

The interval $\mathbb{I}$

The interval $1 + 1 \rightarrowtail \mathbb{I}$ in cSet provides, for every object X:

- a **cylinder**

$$X + X \cong (1 + 1) \times X \rightarrowtail \mathbb{I} \times X \,.$$

- a **path object**

$$X^{\mathbb{I}} \twoheadrightarrow X^{(1+1)} \cong X \times X \,.$$

Moreover $\mathbb{I}$ is **tiny**, $(-) \times \mathbb{I} \dashv (-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$.

The Quillen Model Structure

The classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ can be described succinctly as:

- the **cofibrations** $\mathcal{C}$ are an axiomatized class of monos,
- the **fibrations** $\mathcal{F}$ are those $f : Y \to X$ for which the *gap maps*

$$(\delta \Rightarrow f) : Y^{\mathbb{I}} \longrightarrow X^{\mathbb{I}} \times_X Y$$

  lift on the right against all cofibrations,
- the **weak equivalences** $\mathcal{W}$ are then determined.
  (They can be shown to be those $f : X \to Y$ for which
  $K^f : K^Y \to K^X$ is bijective under π_0 whenever K is fibrant.)

The Quillen Model Structure

The verification of the axioms proceeds in three steps

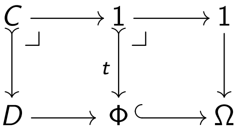
1. a classifier $\Phi \hookrightarrow \Omega$ for **cofibrations** is used to determine a weak factorization system $(\mathcal{C}, \mathsf{TFib})$,
2. the interval $1 \rightrightarrows \mathbb{I}$ is then used to determine the **fibrations** and a second weak factorization system $(\mathsf{TCof}, \mathcal{F})$,
3. the **weak equivalences** are defined by setting

$$\mathcal{W} = \mathsf{TFib} \circ \mathsf{TCof},$$

and the 2-of-3 condition is shown by first constructing a **univalent universe** $\dot{\mathsf{U}} \to \mathsf{U}$.

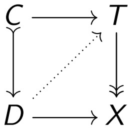
1. The cofibration wfs $(\mathcal{C}, \mathrm{TFib})$

The **cofibrations** $\mathcal{C}$ are the monos $C \rightarrowtail D$ classified by $t : 1 \rightarrowtail \Phi$.



The **trivial fibrations** TFib are the maps $T \twoheadrightarrow X$ that lift against the cofibrations.

$$\mathcal{C}^{\pitchfork} =: \mathrm{TFib}$$



1. The cofibration wfs $(\mathcal{C}, \mathsf{TFib})$

Proposition

$(\mathcal{C}, \mathsf{TFib})$ *is an algebraic weak factorization system.*

Proof.

The classifier $t : 1 \rightarrowtail \Phi$ determines a fibered polynomial monad

$$P_t = \Phi_! t_* : \mathsf{cSet} \longrightarrow \mathsf{cSet}$$

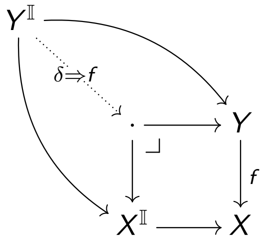
the algebras for which in $\mathsf{cSet}/_X$ are the trivial fibrations. $\square$

2. The fibration wfs $(\mathsf{TCof}, \mathcal{F})$

The **fibrations** $\mathcal{F}$ are defined in terms of the trivial fibrations by

$$(f : Y \to X) \in \mathcal{F} \qquad \text{iff} \qquad (\delta \Rightarrow f) \in \mathsf{TFib}$$

for both **gap maps** $\delta \Rightarrow f$ for the endpoints $\delta : 1 \longrightarrow \mathbb{I}$.



The **trivial cofibrations** TCof are the maps that lift against $\mathcal{F}$.

$$\mathsf{TCof} := {}^{\pitchfork}\mathcal{F}$$

3. The weak equivalences $\mathcal{W}$

Proposition

Let $\mathcal{W} := \mathsf{TFib} \circ \mathsf{TCof}$*. Then*

$$\mathsf{TCof} = \mathcal{W} \cap \mathcal{C}$$
$$\mathsf{TFib} = \mathcal{W} \cap \mathcal{F}$$

so $(\mathcal{C}, \mathsf{TFib})$ *and* $(\mathsf{TCof}, \mathcal{F})$ *form a Barton* **premodel structure***.*

Corollary

If $\mathcal{W}$ *satisfies 2-of-3, then* $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *is a QMS.*

3. The weak equivalences $\mathcal{W}$

We show 2-of-3 for $\mathcal{W}$ using a **universal fibration** $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$.

(i) there is a **universal small map** $\dot{\mathsf{V}} \to \mathsf{V}$,

(ii) U is the **classifying type** for fibration structures on $\dot{\mathsf{V}} \to \mathsf{V}$,

(iii) $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is **univalent**,

(iv) U is **fibrant**,

(v) fibrant U implies **2-of-3** for $\mathcal{W}$.

The idea of getting a QMS *from* univalence is due to Sattler.

3(i). The universal small map $\dot{\mathsf{V}} \to \mathsf{V}$

The **category of elements** functor $\int_{\mathbb{C}}$

$$\textstyle\int_{\mathbb{C}} : \widehat{\mathbb{C}} \rightleftarrows \mathsf{Cat} : \nu_{\mathbb{C}}$$

always has a right adjoint **nerve** functor $\nu_{\mathbb{C}}$.

Proposition

For any small map $Y \to X$ in $\widehat{\mathbb{C}}$ there is a canonical pullback

$$\begin{array}{ccc} Y & \longrightarrow & \nu_{\mathbb{C}}\, \dot{\mathsf{set}}^{\mathsf{op}} \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & \nu_{\mathbb{C}}\, \mathsf{set}^{\mathsf{op}} \end{array}$$

since $\dot{\mathsf{set}}^{\mathsf{op}} \longrightarrow \mathsf{set}^{\mathsf{op}}$ *classifies small discrete fibrations in* Cat.

3(i). The universal small map $\dot{\mathsf{V}} \to \mathsf{V}$

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$$\begin{array}{ccccc} Y & \longrightarrow & \nu_{\mathbb{C}}\,\dot{\mathsf{set}}^{\mathsf{op}} & = & \dot{\mathsf{V}} \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ X & \longrightarrow & \nu_{\mathbb{C}}\,\mathsf{set}^{\mathsf{op}} & = & \mathsf{V} \end{array}$$

since $\dot{\mathsf{set}}^{\mathsf{op}} \to \mathsf{set}^{\mathsf{op}}$ *classifies small discrete fibrations in* Cat*.*

3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

Proposition

There is a small fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ *such that every small fibration* $A \twoheadrightarrow X$ *has a classifying map* $a : X \to \mathsf{U}$ *fitting into a pullback*

$$\begin{array}{ccc} A & \longrightarrow & \dot{\mathsf{U}} \\ \big\downarrow\!\!\downarrow & \lrcorner & \big\downarrow\!\!\downarrow \\ X & \xrightarrow[a]{} & \mathsf{U} \end{array}$$

3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

Definition

A *universal fibration* is a small fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ such that every small fibration $A \twoheadrightarrow X$ is a pullback of $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ along a *classifying* map $X \to \mathsf{U}$.

$$\begin{array}{ccc} A & \longrightarrow & \dot{\mathsf{U}} \\ \big\downarrow\!\!\!\downarrow & \lrcorner & \big\downarrow\!\!\!\downarrow \\ X & \longrightarrow & \mathsf{U} \end{array}$$

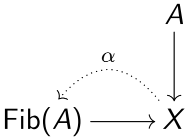
We will construct a universal fibration using the classifying type for fibration structures.

3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

For any $A \to X$ there is a *classifying type for fibration structures*,

$$\mathsf{Fib}(A) \longrightarrow X,$$

sections of which correspond to fibration structures α on $A \to X$.



NB: $\mathsf{Fib}(A) \to X$ is small when $A \to X$ is small.

3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

The map $\mathsf{Fib}(A) \to X$ is stable under pullback,

$$f^*\mathsf{Fib}(A) \cong \mathsf{Fib}(f^*A).$$

Thus the bottom square below is also a pullback.

$$\begin{array}{ccc}
f^*A & \longrightarrow & A \\
\downarrow & & \downarrow \\
Y & \xrightarrow[f]{} & X \\
\uparrow & & \uparrow \\
\mathsf{Fib}(f^*A) & \longrightarrow & \mathsf{Fib}(A)
\end{array}$$

The construction of $\mathsf{Fib}(A)$ uses the **Frobenius** condition, as well as the **root functor** $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$. This is where we use the fact that the interval $\mathbb{I}$ is **tiny**.

3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

Now let U be the type of fibration structures on $\dot{\mathsf{V}} \to \mathsf{V}$,

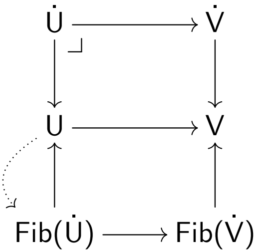
$$\mathsf{U} = \mathsf{Fib}(\dot{\mathsf{V}}) \longrightarrow \mathsf{V}.$$

Then define $\dot{\mathsf{U}} \to \mathsf{U}$ by pulling back the universal small map:

$$\begin{array}{ccc} \dot{\mathsf{U}} & \longrightarrow & \dot{\mathsf{V}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathsf{U} & \longrightarrow & \mathsf{V} \end{array}$$

3(ii). The universal fibration $\dot{U} \twoheadrightarrow U$

Since Fib(−) is stable under pullback, the lower square below is a pullback.

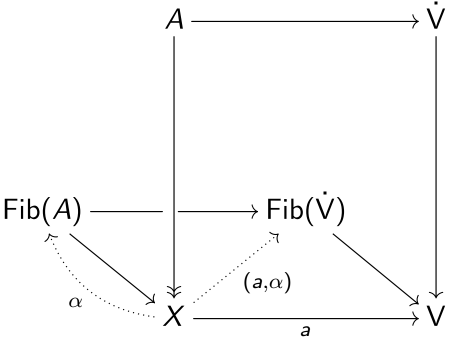


Since U = Fib($\dot{V}$), there is a section of Fib($\dot{U}$) (namely Δ_U).

So $\dot{U} \to U$ is a fibration.

3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

A fibration structure α on a small map $A \to X$ then gives rise to a factorization (a, α) of its classifying map $a : X \to \mathsf{V}$.



3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

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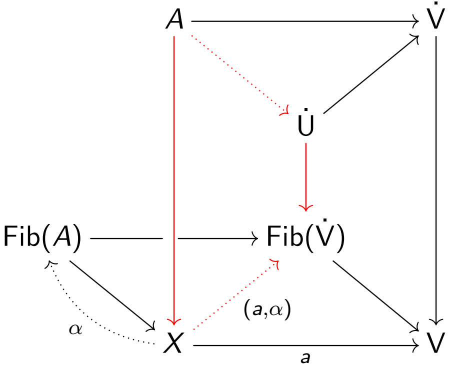


which then classifies it as a fibration, since $\mathsf{Fib}(\dot{\mathsf{V}}) = \mathsf{U}$.

3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ in type theory

The type of fibration structures $\mathsf{Fib}(A)$ is an example of type-theoretic thinking.

It can be constructed as the "type of proofs that A is a fibration" using the **propositions-as-types** idea.

A fibration on X is then a pair (A, α) consisting of a small family $A : X \to \mathsf{V}$ together with a **proof** $\alpha : \mathsf{Fib}(A)$ that A is a fibration.

The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is therefore

$$\mathsf{U} = \sum_{A:\mathsf{V}} \mathsf{Fib}(A)\,,$$

$$\dot{\mathsf{U}} = \sum_{(A,\alpha):\mathsf{U}} A\,.$$

3(iii). $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is univalent

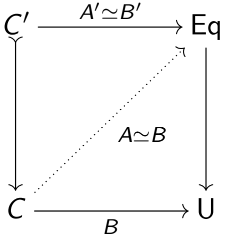
The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is **univalent** if the type of (based) equivalences $\mathsf{Eq} \to \mathsf{U}$ is a trivial fibration.

(Once we have the QMS this will imply

$$\mathsf{Id}(A, B) \simeq \mathsf{Eq}(A, B)$$

by the interpretation of Id_{U} as the pathspace $\mathsf{U}^{\mathbb{I}}$.)

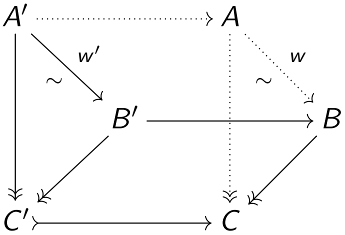
That $\mathsf{Eq} \to \mathsf{U}$ is in TFib means it has the RLP against $\mathcal{C}$:



3(iii). $\dot{\mathrm{U}} \twoheadrightarrow \mathrm{U}$ is univalent

Definition (EEP)

The **equivalence extension property** says that weak equivalences extend along cofibrations $C' \rightarrowtail C$ as follows: given fibrations $A' \twoheadrightarrow C'$ and $B \twoheadrightarrow C$ and a weak equivalence $w' : A' \simeq B'$, where $B' = C' \times_C B$,



there is a fibration $A \twoheadrightarrow C$ and a weak equivalence $w : A \simeq B$, which pulls back to w'.

3(iii). $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is univalent

Voevodsky proved this for simplicial sets and Kan fibrations, to give the following.

Theorem (Voevodsky)

There is a universal small Kan fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ *in simplicial sets that is univalent.*

Coquand later gave a constructive proof for cubical sets, using type theoretic reasoning.

We have adapted Coquand's proof to a new homotopical one that holds in many QMCs (without using 2-of-3).

3(iv). U is fibrant

From univalence, we can show that the base object U is fibrant.

Theorem

The universe U *is fibrant.*

Voevodsky proved this directly for Kan simplicial sets using *minimal fibrations*, which are specific to that setting.

Shulman gave a general proof from univalence, but it uses 2-of-3 for $\mathcal{W}$, and so cannot be used here.

Coquand gave a proof from univalence that avoids 2-of-3, using a type theoretic reduction of fibrancy to *Kan composition*.

We have a new general proof from univalence that avoids 2-of-3.

3(iv). U is fibrant

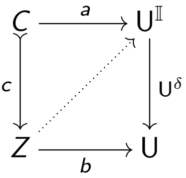
It suffices to show:

Proposition

The evaluation at an endpoint $\mathsf{U}^{\mathbb{I}} \longrightarrow \mathsf{U}$ *is a trivial fibration.*

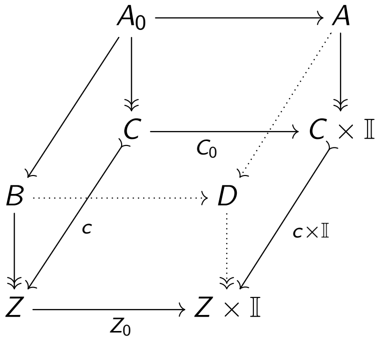
Proof.

We need to solve the following filling problem for any cofibration c.



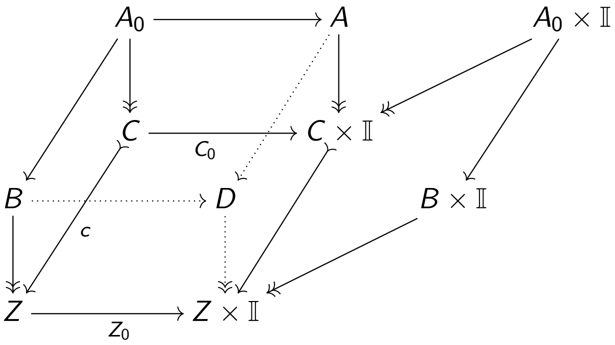
3(iv). U is fibrant

Transposing by $\mathbb{I}$ and using the classifying property of U gives the following equivalent problem.



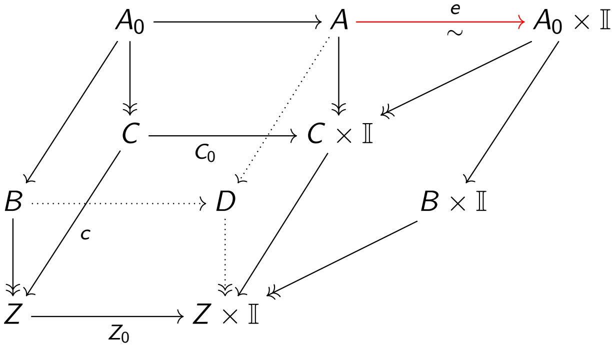
3(iv). U is fibrant

Now apply the functor $(-) \times \mathbb{I}$ to the left face to get:



3(iv). U is fibrant

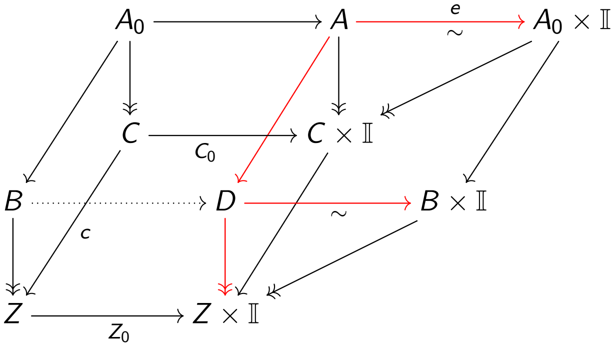
Now apply the functor $(-) \times \mathbb{I}$ to the left face to get:



There is a weak equivalence $e : A \xrightarrow{\sim} A_0 \times \mathbb{I}$, to which we can apply the EEP.

3(iv). U is fibrant

Now apply the functor $(-) \times \mathbb{I}$ to the left face to get:



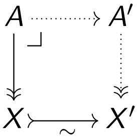
There is a weak equivalence $e : A \simeq A_0 \times \mathbb{I}$, to which we can apply the EEP. This produces the required fibration $D \twoheadrightarrow Z \times \mathbb{I}$. □

3(v). From U fibrant to 2-of-3 for $\mathcal{W}$

Finally, we can apply the following.

Proposition (Sattler)

The weak equivalences satisfy 2-of-3 if the fibrations extend along the trivial cofibrations.

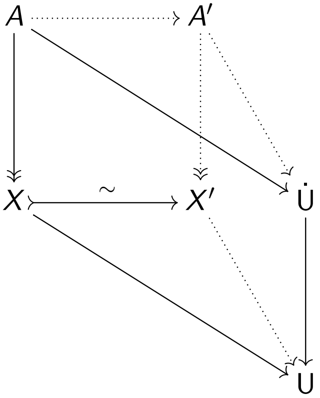


This is called the **fibration extension property**.

3(v). From U fibrant to 2-of-3 for $\mathcal{W}$

Lemma

Given a universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$, the FEP holds if U is fibrant.



□

Danke!

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