Algebraic Type Theory

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Outline

1. Strictifying Homotopical Models

- A **homotopical model** of (homotopy) type theory was defined to be a Quillen model category $\mathcal{E}$, with the Frobenius property and a fibrant, univalent universe $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$.
- We can extract a **strict model** of (homotopy) type theory from a homotopical one using some ideas of Voevodsky and Lumsdaine-Warren.
- The resulting structure is a **category with families**, which is a quite strict notion of a model of dependent type theory. (A related construction gives a **contextual category**.)
- I will do the cases of Σ and Π types, but one can add the Id-types and a universe U.

1. Dependent type theory

The system to be modelled has:

Basic types and terms: $A, B, \ldots, x{:}A, b{:}B, \ldots$

Dependent types and terms: $x{:}A \vdash b(x) : B(x), \ldots$

Contexts: $(x{:}A, y{:}B(x), \ldots), \Gamma, \Delta, \ldots$

Substitutions: $\sigma{:}\Delta \to \Gamma, \ldots$

Type forming operations: $\sum_{x:A} B(x), \prod_{x:A} B(x), \ldots$

Equations between terms: $\Gamma \vdash s = t : A$

1. Dependent type theory: Rules

Contexts:

$$\frac{x : A \vdash B(x)}{x : A,\ y : B(x) \vdash} \qquad \frac{\Gamma \vdash C}{\Gamma, z : C \vdash}$$

Sums:

$$\frac{x : A \vdash B(x)}{\sum_{x:A} B(x)} \qquad \frac{a : A \qquad b : B(a)}{\langle a, b \rangle : \sum_{x:A} B(x)}$$

$$\frac{c : \sum_{x:A} B(x)}{\mathsf{fst}\, c : A} \qquad \frac{c : \sum_{x:A} B(x)}{\mathsf{snd}\, c : B(\mathsf{fst}\, c)}$$

$$\mathsf{fst}\langle a, b \rangle = a : A \qquad \mathsf{snd}\langle a, b \rangle = b : B$$

$$\langle \mathsf{fst}\, c, \mathsf{snd}\, c \rangle = c : \sum_{x:A} B(x)$$

1. Dependent type theory: Rules

Products:

$$\frac{x : A \vdash B(x)}{\prod_{x:A} B(x)} \qquad \frac{x : A \vdash b : B(x)}{\lambda x.b : \prod_{x:A} B(x)}$$

$$\frac{a : A \qquad f : \prod_{x:A} B(x)}{fa : B(a)}$$

$$x : A \vdash (\lambda x.b)x = b : B(x)$$

$$\lambda x.fx = f : \prod_{x:A} B(x)$$

Substitution:

$$\frac{\sigma : \Delta \to \Gamma \qquad \Gamma \vdash a : A}{\Delta \vdash a[\sigma] : A[\sigma]}$$

2. Natural Models of Type Theory

Definition

A natural transformation $f : Y \to X$ of presheaves on a category $\mathbb{C}$ is called **representable** if its pullback along any $\mathrm{y}C \to X$ is represented:

$$\begin{array}{ccc} \mathrm{y}D & \longrightarrow & Y \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\ \mathrm{y}C & \longrightarrow & X \end{array}$$

Proposition

A representable natural transformation is the same thing as a **category with families** *in the sense of Dybjer.*

2. Natural Models as CwFs

Write the objects and arrows of $\mathbb{C}$ as $\sigma : \Delta \to \Gamma$, giving the **category of contexts and substitutions**.

A CwF is usually defined as a presheaf of **types in context**,

$$\mathsf{Ty} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}\,,$$

together with a presheaf of **typed terms**,

$$\mathsf{Tm} : (\textstyle\int_{\mathbb{C}} \mathsf{Ty})^{\mathrm{op}} \to \mathsf{Set}\,.$$

But we will reformulate this notion using the equivalence

$$\mathsf{Set}^{(\int_{\mathbb{C}} \mathsf{Ty})^{\mathrm{op}}} \;\simeq\; \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}/\mathsf{Ty}\,.$$

So we will instead have a map $p : \mathsf{Tm} \to \mathsf{Ty}$.

2. Natural Models as CwFs

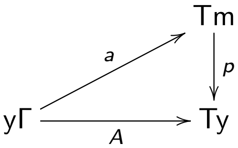
Let $p : \mathsf{Tm} \to \mathsf{Ty}$ be a **representable** map of presheaves on $\mathbb{C}$.

Then Ty is again the **presheaf of types in context**, and now Tm is the **presheaf of terms in context**, and p gives the **typing of terms**.

Formally, we **interpret**:

$$\begin{array}{rcl} \Gamma \vdash A & \approx & A \in \mathsf{Ty}(\Gamma) \\ \Gamma \vdash a : A & \approx & a \in \mathsf{Tm}(\Gamma) \end{array}$$

where $A = p \circ a$.

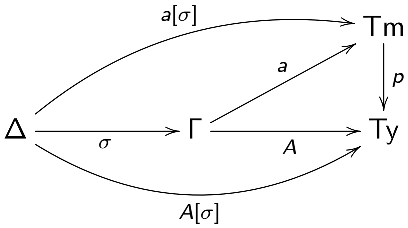


NB: we will now just write Γ rather than yΓ for the representables.

2. Natural Models as CwFs

Naturality of $p : \mathsf{Tm} \to \mathsf{Ty}$ means that for any **substitution** $\sigma : \Delta \to \Gamma$, we have the required action on types and terms:

$$\begin{aligned} \Gamma \vdash A \quad &\Rightarrow \quad \Delta \vdash A[\sigma] \\ \Gamma \vdash a : A \quad &\Rightarrow \quad \Delta \vdash a[\sigma] : A[\sigma] \end{aligned}$$



2. Natural Models as CwFs

Given any further $\tau : \Delta' \to \Delta$ we clearly have

$$A[\sigma][\tau] = A[\sigma \circ \tau] \qquad\qquad a[\sigma][\tau] = a[\sigma \circ \tau]$$

and for the identity substitution $1 : \Gamma \to \Gamma$ we have

$$A[1] = A \qquad\qquad a[1] = a\,.$$

This is the **basic structure** of a CwF.

The remaining operation of **context extension**

$$\frac{\Gamma \vdash A}{\Gamma, x\!:\!A \vdash}$$

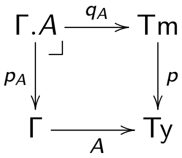
is given by the representability of $p : \mathsf{Tm} \to \mathsf{Ty}$ as follows.

2. Natural Models: Context Extension

Given $\Gamma \vdash A$ we need a new context $\Gamma.A$ together with a substitution $p_A : \Gamma.A \to A$ and a term

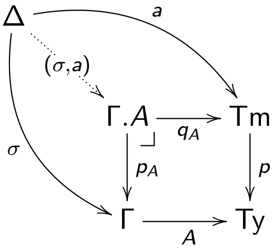
$$\Gamma.A \vdash q_A : A[p_A]\,.$$

Let $p_A : \Gamma.A \to \Gamma$ be the pullback of p along A.



The map $q_A : \Gamma.A \to \mathrm{Tm}$ gives the required term $\Gamma.A \vdash q_A : A[p_A]$.

2. Natural Models: Context Extension



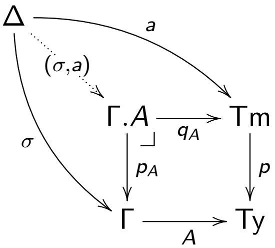
The pullback means that given any substitution $\sigma : \Delta \to \Gamma$ and term $\Delta \vdash a : A[\sigma]$ there is a map

$$(\sigma, a) : \Delta \to \Gamma.A$$

satisfying

$$p_A(\sigma, a) = \sigma$$
$$q_A[\sigma, a] = a.$$

2. Natural Models: Context Extension



By the uniqueness of (σ, a), we also have

$$(\sigma, a) \circ \tau \;=\; (\sigma \circ \tau, a[\tau]) \qquad \text{for any } \tau : \Delta' \to \Delta$$

and

$$(p_A, q_A) = 1.$$

These are precisely the **laws of a CwF**, under the equivalence

$$\mathsf{Set}^{(\int_{\mathbb{C}} \mathsf{Ty})^{\mathrm{op}}} \simeq \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}/\mathsf{Ty}$$

□

2. Natural Models and Clans

Let $p : \dot{\mathsf{U}} \to \mathsf{U}$ be a natural model.

The fibration $\mathcal{F}_p \to \mathbb{C}$ of all pullbacks of p

$$A^*p : \Gamma.A \to \Gamma \qquad\qquad \text{for all } A : \Gamma \to \mathsf{U}$$

form a **display map category** (=: **pre-clan**).

Conversely, given any pre-clan $(\mathbb{C}, \mathcal{F})$, there is a natural model $p_{\mathcal{F}} : \dot{\mathsf{U}}_{\mathcal{F}} \to \mathsf{U}_{\mathcal{F}}$ over $\mathbb{C}$,

$$p_{\mathcal{F}} = \coprod_{f \in \mathcal{F}} \mathsf{y}f : \coprod_{f \in \mathcal{F}} \mathsf{y}\mathrm{dom}(f) \to \coprod_{f \in \mathcal{F}} \mathsf{y}\mathrm{cod}(f)\,.$$

There is an adjunction $p \dashv \mathcal{F}$

$$\mathsf{PreClan} \underset{p}{\overset{\mathcal{F}}{\rightleftarrows}} \mathsf{NatMod}\,.$$

2. Natural Models and Initiality

- The notion of a natural model is **essentially algebraic** (generalized algebraic, dependently typed algebraic, clan algebraic, finite limit theory, ...).
- The algebraic homomorphisms correspond exactly to **syntactic translations**.
- There are **initial algebras**, as well as **free algebras** over basic types and terms.
- The rules of type theory can be seen as a procedure for **generating the free algebras**.

3. Modeling the Type Formers

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ determines a **polynomial endofunctor**

$$P : \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}} \to \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}},$$

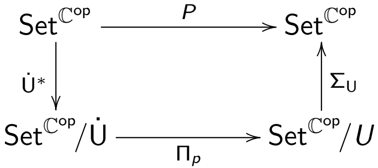
defined for every $X : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ by

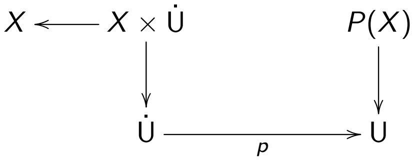
$$P(X) \;=\; \sum_{A\,:\,\mathsf{U}} X^{[A]}\,,$$

where $[A] = p^{-1}(A)$ is the fiber of $p : \dot{\mathsf{U}} \to \mathsf{U}$ at $A : \mathsf{U}$.

3. Modeling the Type Formers: Polynomials

The construction of $P(X)$ can be described as follows.

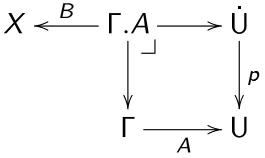




3. Modeling the Type Formers

Lemma (UMP of polynomials)

Maps $\Gamma \to P(X)$ *correspond naturally to pairs* (A, B) *where:*

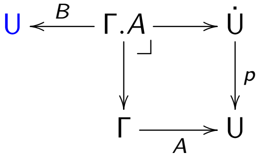


3. Modeling the Type Formers

Applying P to U itself therefore gives an object

$$P(\mathsf{U}) = \sum_{A:\mathsf{U}} \mathsf{U}^{[A]}$$

such that maps $\Gamma \to P(\mathsf{U})$ correspond naturally to **types in an extended context** $\Gamma.A \vdash B$



3. Modeling the Type Formers: Π

Proposition

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for Π-types just if there are maps λ, Π making the following a pullback.

$$\begin{array}{ccc} P(\dot{\mathsf{U}}) & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ {\scriptstyle P(p)}\downarrow & & \downarrow{\scriptstyle p} \\ P(\mathsf{U}) & \xrightarrow[\Pi]{} & \mathsf{U} \end{array}$$

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Proof:

$$\begin{array}{ccc} P(\dot{\mathsf{U}}) & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ \downarrow & & \downarrow{\scriptstyle p} \\ P(\mathsf{U}) & \xrightarrow[\Pi]{} & \mathsf{U} \end{array}$$

$$\sum_{A:\mathsf{U}} \mathsf{U}^{[A]}$$

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$A \vdash B$ $\qquad\qquad\qquad$ $\Pi_A B$

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Proof:

$A \vdash b : B$ $\qquad\qquad$ $\lambda_A b$

$$\begin{array}{ccccc}
\sum_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]} & & P(\dot{\mathsf{U}}) & \xrightarrow{\ \lambda\ } & \dot{\mathsf{U}} \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle p} \\
\sum_{A:\mathsf{U}} \mathsf{U}^{[A]} & & P(\mathsf{U}) & \xrightarrow[\ \Pi\ ]{} & \mathsf{U}
\end{array}$$

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Proof:

$$\begin{array}{ccccc}
 & & & & f \\
\sum_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]} & & P(\dot{\mathsf{U}}) & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\
 & & \downarrow & & \downarrow p \\
\sum_{A:\mathsf{U}} \mathsf{U}^{[A]} & & P(\mathsf{U}) & \xrightarrow[\Pi]{} & \mathsf{U} \\
A \vdash B & & & & \Pi_A B
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Proposition

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for Π-types just if there are maps λ, Π making the following a pullback.

Proof:

$$A \vdash f(x) : B \qquad\qquad \lambda_A f(x) = f$$

$$\begin{array}{ccccc}
\sum\limits_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]} & & P(\dot{\mathsf{U}}) & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\
 & & \downarrow & & \downarrow p \\
\sum\limits_{A:\mathsf{U}} \mathsf{U}^{[A]} & & P(\mathsf{U}) & \xrightarrow[\Pi]{} & \mathsf{U}
\end{array}$$

$$A \vdash B \qquad\qquad \Pi_A B$$

3. Modeling the Type Formers: Σ

Proposition

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ *models the rules for* Σ*-types just if there are maps* (pair, Σ) *making the following a pullback*

$$\begin{array}{ccc} Q & \xrightarrow{\ \mathsf{pair}\ } & \dot{\mathsf{U}} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ P(\mathsf{U}) & \xrightarrow[\ \Sigma\ ]{} & \mathsf{U} \end{array}$$

where $q : Q \to P(\mathsf{U})$ *is the polynomial composition* $P_q = P \circ P$.

Explicitly:

$$Q \ = \ \sum_{A:\mathsf{U}} \sum_{B:\mathsf{U}^A} \sum_{x:A} B(x)$$

3. Modeling the Type Formers: Strictification

Theorem

Given any Π-**tribe** $(\mathbb{C}, \mathcal{F})$, *for example a Quillen model category with the Frobenius property, the associated natural model* $p_{\mathcal{F}} : \dot{\mathsf{U}}_{\mathcal{F}} \to \mathsf{U}_{\mathcal{F}}$ *under the adjunction*

$$\mathsf{PreClan} \underset{p}{\overset{\mathcal{F}}{\rightleftarrows}} \mathsf{NatMod}\,.$$

has Σ *and* Π *types (as well as* Id*-types).*

The natural model $p_{\mathcal{F}} : \dot{\mathsf{U}}_{\mathcal{F}} \to \mathsf{U}_{\mathcal{F}}$ is thus a **strictification** of the homotopical model $(\mathbb{C}, \mathcal{F})$.

4. A Polynomial Monad

Consider the rules for a terminal type T.

$$\frac{}{\vdash \mathsf{T}} \qquad\qquad \frac{}{\vdash * : \mathsf{T}} \qquad\qquad \frac{}{x : \mathsf{T} \vdash x = * : \mathsf{T}}$$

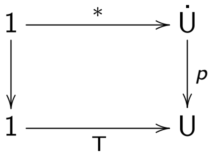
Proposition

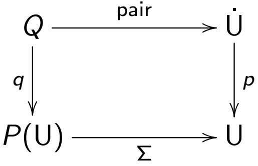
A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for a terminal type just if there are maps $(*, \mathsf{T})$ making the following a pullback.

$$\begin{array}{ccc} 1 & \xrightarrow{\;*\;} & \dot{\mathsf{U}} \\ \downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[\;\mathsf{T}\;]{} & \mathsf{U} \end{array}$$

4. A Polynomial Monad

Consider the pullback squares for T and Σ.





These determine cartesian natural transformations between the corresponding polynomial endofunctors.

$$\tau : 1 \Rightarrow P \qquad\qquad \sigma : P \circ P \Rightarrow P$$

4. A Polynomial Monad

Theorem (A-Newstead)

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models T *and* Σ *types just if the associated polynomial endofunctor* P *has the structure of a cartesian* **monad**.

$$\tau : 1 \Rightarrow P \qquad\qquad \sigma : P \circ P \Rightarrow P$$

4. A Polynomial Monad

The **monad laws** correspond to the following type isomorphisms.

$\sigma \circ P\sigma = \sigma \circ \sigma_P$	$\sum_{a:A} \sum_{b:B(a)} C(a,b) \;\cong\; \sum_{(a,b):\sum_{a:A} B(a)} C(a,b)$
$\sigma \circ P\tau = 1$	$\sum_{a:A} 1 \;\cong\; A$
$\sigma \circ \tau_P = 1$	$\sum_{x:1} A \;\cong\; A$

4. A Polynomial Monad

The pullback square for Π

$$\begin{array}{ccc} P(\dot{\mathsf{U}}) & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ {\scriptstyle P(p)}\downarrow & & \downarrow{\scriptstyle p} \\ P(\mathsf{U}) & \xrightarrow[\Pi]{} & \mathsf{U} \end{array}$$

determines a cartesian natural transformation

$$\pi : P^{\mathbb{2}}(p) \Rightarrow p$$

where $P^{\mathbb{2}} : \hat{\mathbb{C}}^{\mathbb{2}} \to \hat{\mathbb{C}}^{\mathbb{2}}$ is the lift of P to the arrow category $\hat{\mathbb{C}}^{\mathbb{2}}$.

4. A Polynomial Monad

So a natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models Π types just if it has an **algebra structure** for the lifted endofunctor $P^{\mathbb{2}}$.

$$\pi : P^{\mathbb{2}}(p) \Rightarrow p$$

The **algebra laws** correspond to the following type isomorphisms.

$\pi \circ P\pi \;=\; \pi \circ \sigma$	$\prod_{a:A} \prod_{b:B(a)} C(a,b) \;\cong\; \prod_{(a,b):\sum_{a:A} B(a)} C(a,b)$
$\pi \circ \tau \;=\; 1$	$\prod_{x:1} A \;\cong\; A$

Summary: Strictification

Theorem

A homotopical model of (homotopy) type theory gives rise to a natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ *which*

(i) presents a polynomial monad, and an algebra for it, and

(ii) strictly models the Π, Σ *and* Id *type formers.*

References

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