Limits and exponentiable functors in synthetic ∞-categories

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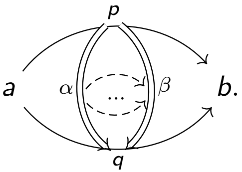
Outline

∞-categories

- Idea of an ∞-category: It is what follows in 1-category, 2-category, ..., n-category ...
- Roughly (∞, n)-category looks something like



- Problem: Many definitions, and none of them is easy, some of them remain incomparable.
- Possible solution (for $(\infty, 1)$-categories): Synthetic theory using type theory (Riehl-Shulman [RS17]) or set-theoretic (Riehl-Verity [RV22]). And more recently (also type theoretic) Cisinski-Nguyen-Walde.
- Our contribution: Theory of (Co)Limits and exponentiable functors [BM22].

Simplicial HoTT

- We start with HoTT:
  - $*$ Dependent types; $B : A \to \mathcal{U}$.
  - $*$ Dependent sums; $\sum$.
  - $*$ Dependent products; $\prod$.
  - $*$ Univalence axiom.
- Buchholtz-Weinberger in [BW21] observed that sHoTT can be obtained by adding a strict interval to HoTT:
  A type $\mathbb{2}$ with distinct endpoints $0,\ 1$ with an order relation making it into a strict interval.

- One feature of sHoTT that simplify it are *shapes* and *subshapes*.
- We can build simplicies:

$$\Delta^n := \{(t_1, t_2, \ldots, t_n) : \mathbb{2}^n \mid t_n \leq \ldots t_2 \leq t_1\}$$

  And also $\partial\Delta^n$, Λ^n_k for $0 \leq k \leq n$.
- Another ingredient is **extension types**:
  ϕ and ψ are shapes with $\phi \subseteq \psi$, a type family $A : \Gamma \times \psi \to \mathcal{U}$ and $a : \prod_{\Gamma \times \phi} A$ then we form

$$\left\langle \prod_{\Gamma \times \psi} A \mid^{\phi}_{a} \right\rangle := \left\langle \begin{array}{ccc} \phi & \xrightarrow{a} & A \\ \downarrow & \nearrow & \\ \psi & & \end{array} \right\rangle$$

For a type A, we can define:

- Given $x, y : A$, $\hom_A(x, y) := \left\langle \Delta^1 \to A \Big|^{\partial\Delta^1}_{[x,y]} \right\rangle$.
  If $f : \hom(x, y)$ then $f(0) \equiv x$ and $f(1) \equiv y$.
- For $x : A$, an identity arrow $\mathrm{id}_x := \lambda(t : \Delta^1).x$.
- If $f : \hom_A(x, y)$, $g : \hom_A(y, z)$ and $h : \hom_A(x, z)$:

$$\hom^2_A \left(\begin{array}{ccccc} & & y & & \\ & {}^{f}\diagup & & \diagdown^{g} & \\ x & & \underset{h}{\rule{3em}{0.4pt}} & & z \end{array} \right) := \left\langle \Delta^2 \to A \Big|^{\partial\Delta^2}_{[x,y,z,f,g,h]} \right\rangle .$$

Segal types

Segal types are types with "unique composition". By definition, if the type

$$\sum_{h:\hom_A(x,z)} \hom^2_A\left(\begin{array}{ccc} & y & \\ {}^{f}\nearrow & & \searrow^{g} \\ x & \xrightarrow[h]{} & z \end{array}\right)$$

is contractible.

- This is enough to get the categorical structure: we have associative and unital composition.
- Functions $f : A \to B$ between Segal types are "functors".

Theorem (Riehl-Shulman)

The category $\mathbf{sSet}^{\Delta^{op}}$ *supports an interpretation of sHoTT where Segal types correspond to Segal spaces and Rezk types to Rezk spaces.*

Rezk types

- An arrow $f : \hom_A(x, y)$ is an **isomorphism** if
$$\left(\sum_{g:\hom_A(y,x)} g \circ f = \mathrm{id}_x \right) \times \left(\sum_{h:\hom_A(y,x)} f \circ h = \mathrm{id}_y \right).$$
- The type above is a proposition:
$$(x \cong y) := \sum_{f:\hom_A(x,y)} \mathrm{isiso}(f).$$
- By path induction
$$\mathrm{idtoiso} : \prod_{x,y:A} (x = y) \to (x \cong y).$$

A Segal type is **Rezk** if idtoiso is an equivalence.

Adjunctions

A **natural transformation** is an element $\alpha : \hom_{A \to B}(f, g)$.

- Component-wise determined

$$\hom_{A \to B}(f, g) \simeq \prod_{a:A} \hom_B(f(a), g(a)).$$

- The type theory makes them natural.

A **quasi-transposing adjunction** between types A, B consist of **functors** $f : A \to B$ and $u : B \to A$ and a family of equivalences

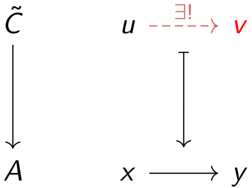
$$\phi : \prod_{a:A, b:B} (\hom_B(fa, b) \simeq \hom_A(a, ub)).$$

Covariant families

A family $C : A \to \mathcal{U}$ is **covariant** if for each $x, y : A$, $f : \hom_A(x, y)$ and $u : C(x)$ the type

$$\sum_{v : C(y)} \hom_{C(f)}(u, v)$$

is contractible.



Theorem (Riehl-Shulman)

If $B : A \to \mathcal{U}$ is a covariant family type over a Segal type A then $\sum_{a:A} B(a)$ is a Segal type.

Theorem

If A is a Rezk type and $C : A \to \mathcal{U}$ is a covariant family over A then $\sum\limits_{x:A} C(x)$ is also a Rezk type.

Outline

(Co)Limits

An element $b : B$ is **initial** if for all $x : B$, $\hom_B(b, x)$ is contractible.
Also, an element $b : B$ is **terminal** if for all $x : B$, $\hom_B(x, b)$ is contractible.
Define the type of **(co)cones** of $f : A \to B$

$$\mathrm{cocone}(f) :\equiv \sum_{b:B} \hom_{B^A}(f, \triangle b) \text{ and } \mathrm{cone}(f) :\equiv \sum_{b:B} \hom_{B^A}(\triangle b, f).$$

Definition

A **colimit** for $f : A \to B$ is an initial element of the type:

$$\mathrm{cocone}(f) :\equiv \sum_{x:B} \hom_{B^A}(f, \triangle x).$$

A **limit** for $f : A \to B$ is an terminal element of the type:

$$\mathrm{cone}(f) :\equiv \sum_{x:B} \hom_{B^A}(\triangle x, f).$$

Classical results (sHoTT version)

Theorem

1. *(Co)Limits are unique up to isomorphism if they exist.*
2. *There exist a colimit (a, α) for f if and only if*

$$\prod_{x:B} (\hom_B(a, x) \simeq \hom_{B^A}(f, \triangle x)).$$

In the presence of an adjunction, then we also have the usual preservation of limits and colimits.

Theorem

Let A, B be Segal types and functions $g : J \to B$, $f : A \to B$, $u : B \to A$, such that g has a limit (b, β) and u is right quasi-transposing adjunction of f then $(u(b), u\beta)$ is a limit for $ug : J \to A$.

Special case: Rezk types

Let B is a Segal type a $f : A \to B$ a function, we define the type:

$$\text{colimit}(f) := \sum_{w:\text{conunder}(f)} \text{isinitial}(w),$$

$$\text{limit}(f) := \sum_{w:\text{cone}(f)} \text{isterminal}(w).$$

Proposition

Over Rezk types, the types colimit(f) and limit(f) are propositions. Therefore, (co)limits are unique.

Outline

Limit of spaces

Let $\{G_i\}_{i \in I}$ be a family of ∞-groupoids indexed by a set I. Denote by G to the obvious diagram $I \to \infty$-**Gpd** then we would expect a formula

$$lim_I G_i = \prod_{i \in I} G_i$$

to be true.
Problem:There is not something like an ∞-category of ∞-groupoids in sHoTT.
Solution: *univalent covariant families* due to [RCS, Cavallo-Riehl-Sattler].

Covariant univalent families

Fix a covariant family $E : B \to \mathcal{U}$ over a Segal type B and $a, b : B$ we obtain a function

$$\text{arrtofun} : \hom_B(a, b) \to (E(a) \to E(b)).$$

E is **covariant univalent** if for all $a, b : B$ the map arrtofun is an equivalence.
For a type A define

$$\text{issmall}_B(A) := \sum_{b:B} (E(b) \simeq A)$$

The type A is B-**small** if $\text{issmall}_B(A)$.

The type B is regarded as an "∞-category" of "∞-groupoids", it is definable in sHoTT and consistent [RCS, Cavallo-Riehl-Sattler].

Limit as dependant product

Let B a Rezk type, a function $f : D \to B$ and assume that $E : B \to \mathcal{U}$ is a covariant univalent family.

Proposition

If (b_0, σ_0) is the center of contraction of $\mathrm{issmall}_B(\prod_{d:D} E(f(d)))$ then b_0 is the limit for f.

Proposition

If $f : D \to B$ has limit (b_0, α) and there is $b_1 : B$ and $u : E(b_1)$. Then $\prod_{d:D} E(f(d))$ is B-small.

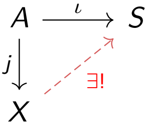
Outline

Segal type completion

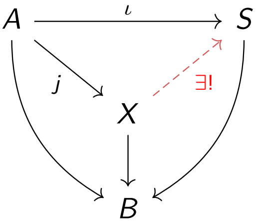
Let A be a type. We say that a Segal type S and a map $\iota : A \to S$ is a **Segal type completion** for A if for any Segal type X, the map

$$\circ\iota : (S \to X) \to (A \to X)$$

is an equivalence. The picture we need is



Furhermore, we need a relative to some fixed Segal type:



Conduché's Theorem in sHoTT

Theorem

Let $f : E \to B$ a map between Segal types, the following are equivalent:

1. *For any Segal type X and $g : X \to B$, the type*

$$\sum_{b:B} (E_b \to X_b)$$

   is Segal.

2. *If $\alpha : \Delta^2 \to B$ and $i : \Lambda_1^2 \to B$, we get the pullbacks*

$$\begin{array}{ccccc} F_2 & \xrightarrow[\iota]{} & F_1 & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow f \\ \Lambda_1^2 & \xrightarrow[i]{} & \Delta^2 & \xrightarrow[\alpha]{} & B. \end{array}$$

   Then F_1 is the Segal type completion of F_2.

Given $f : \mathcal{E} \to \mathcal{B}$ a functor and $a, b \in \mathcal{B}$. We can define a profunctor (bifibration) $hom_{\mathcal{E}} : \mathcal{E}_a^{op} \times \mathcal{E}_b \to \mathbf{Set}$.

Theorem (Conduché)

For a functor $f : \mathcal{E} \to \mathcal{B}$, the following conditions are equivalent:

1. *The functor $f : \mathcal{E} \to \mathcal{B}$ is exponentiable.*
2. *For all $a, b, c \in \mathcal{B}$, $u \in hom_{\mathcal{B}}(a, b)$, $v \in hom_{\mathcal{B}}(b, c)$, $x \in \mathcal{E}_a$, $z \in \mathcal{E}_c$, the induced map*

$$\left(\int^{y \in \mathcal{E}_b} hom^u_{\mathcal{E}}(x, y) \times hom^v_{\mathcal{E}}(y, z) \right) \to hom^{v \circ u}_{\mathcal{E}}(x, z)$$

   is an isomorphism.

For any a, b, $c : B$, $u : \hom_B(a, b)$, $v : \hom_B(b, c)$ and $x : E_a$, $z : E_c$ the map induced by the composition

$$\left(\sum_{y:E_b} \hom_E^u(x, y) \times \hom_E^v(y, z) \right) \to \hom_E^{v \circ u}(x, z)$$

shows $\hom_E^{v \circ u}(x, z)$ as the *discrete type completion* of

$$\sum_{y:E_b} \hom_E^u(x, y) \times \hom_E^v(y, z).$$

One obstruction is that we do not know how (or if it is possible) to prove in sHoTT that *correspondences* $\simeq$ *bifibrations*.

Thank you!

César Bardomiano-Martínez.
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