A Language for Fibered Category Theory

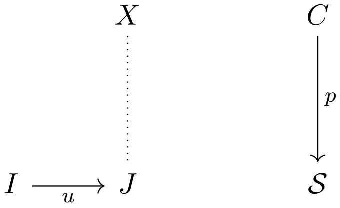
Nico Beck

September 26, 2023

Grothendieck Fibrations

$$\begin{array}{c} C \\ \big\downarrow{\scriptstyle p} \\ \mathcal{S} \end{array}$$

Grothendieck Fibrations



Grothendieck Fibrations

$$
\begin{array}{ccc}
u^*X & \longrightarrow & X \\
\vdots & & \vdots \\
I & \xrightarrow[u]{} & J
\end{array}
\qquad\qquad
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\end{array}
$$

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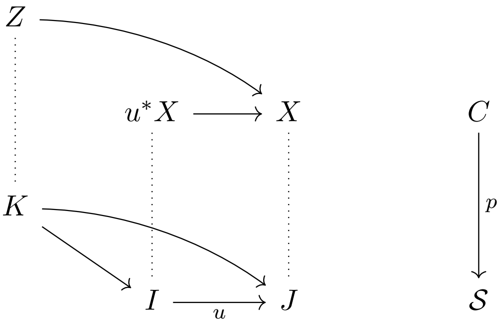

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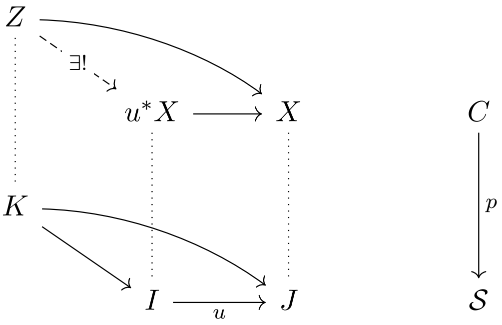

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$$
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\vdots & & \vdots & & \vdots \\
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\end{array}
$$

[1] J. Bénabou, *Fibered categories and the foundations of naive category theory*

Grothendieck Fibrations

From Cat to fibered Cat Theory:

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From Cat to fibered Cat Theory:

- Concepts (completeness, locally small, well-powered,...)
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- Theorems (fibered adjoint functor theorem, relative version of Giraurd's theorem, ...)

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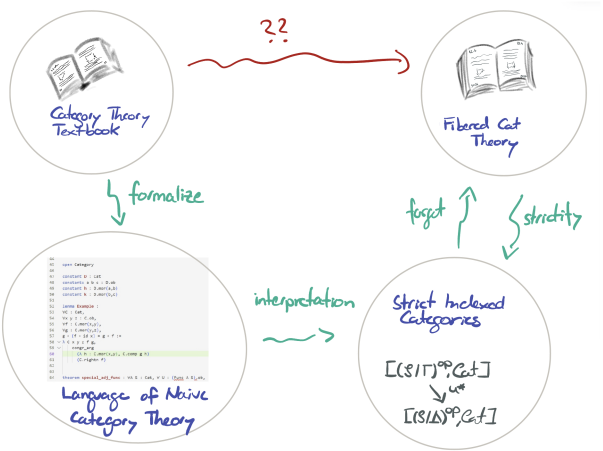
Problem:

It is not easy.

Plan



[2] M. Shulman, Slides: *Large categories and quantifiers in topos theory*

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Examples:

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- Pretopos with coherent topology.
- Grothendieck topos with canonical topology.

[3] M. Shulman, *Stack semantics and the comparison of structural and material set theories*

The Language

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$$\Xi \vdash C : \mathrm{Cat} \qquad \Xi \vdash C \equiv D : \mathrm{Cat}$$

$$\Xi \vdash X : C_0 \qquad \Xi \vdash X \equiv Y : C_0$$

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$$C : \mathrm{Cat}, D : \mathrm{Cat}, c : C_0 \vdash I^0_{\mathrm{func}}(x.c, x.y.f. \mathrm{id}_c) : (C^D)_0$$

[5] M. Makkai, FOLDS

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- Category Constructors.

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$$\frac{\Xi \vdash A, B, C : \mathrm{Cat} \qquad \Xi \vdash F : (C^A)_0 \qquad \Xi \vdash G : (C^B)_0}{\Xi \vdash F/G : \mathrm{Cat}}$$

$$\frac{\Xi \vdash A : \mathrm{Set} \qquad \Xi, x : A \vdash C : \mathrm{Cat}}{\Xi \vdash \prod_{x:A} C : \mathrm{Cat}} \qquad \frac{}{J : \mathrm{Cat}} \text{ (J finite cat)}$$

$$\frac{\Xi \vdash C : \mathrm{Cat} \qquad \Xi \vdash D : \mathrm{Cat}}{\Xi \vdash D^C : \mathrm{Cat}} \qquad \frac{\Xi \vdash C : \mathrm{Cat}}{\Xi \vdash \mathrm{Fam}(C) : \mathrm{Cat}}$$

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3. Choose values for the remaining variables in the large context Λ.
4. Interpret in $[(\mathcal{S}/\Gamma)^{op}, \mathrm{Cat}]$.

$\Gamma \vdash C : \mathrm{Cat}$	$[\![C]\!]$ 0-cell in $[(\mathcal{S}/\Gamma)^{op}, \mathrm{Cat}]$
$\Gamma \vdash X : C_0$	$[\![X]\!]$ object in $[\![C]\!](\mathrm{id}_\Gamma)$
$\Gamma \vdash N : C_1(X, Y)$	$[\![N]\!]$ morphism $[\![X]\!] \to [\![Y]\!]$ in $[\![C]\!](\mathrm{id}_\Gamma)$

[4] M. Shulman, nLab, *2-categorical logic*

How Expressive Is The Language?

Internal: "C is locally small".

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$$C : \mathrm{Cat} \vdash \forall x, y : C_0 . \exists H : \mathrm{Set}_0 . \exists f : (\textstyle\prod_{h:H} C)_1((x)_{h:H}, (y)_{h:H}).$$
$$\forall g : C_1(x, y) . \exists! h : H . f_h = g : \mathrm{Prop}_0$$

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External meaning: "C is locally small in the fibered sense."

How expressive is the language?

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\begin{aligned}
C : \mathrm{Cat} \vdash & \forall A : \mathrm{Set}_0 . \forall X : (\textstyle\prod_{x:A} C)_0 . \\
& \exists P : C_0 . \exists \pi : (\textstyle\prod_{x:A} C)_1 ((P)_{x:A}, (X_x)_{x:A}) . \\
& \forall Y : C_0 . \forall f : (\textstyle\prod_{x:A} C)_1 ((Y)_{x:A}, (X_x)_{x:A}) . \\
& \exists ! g : C_1(Y, P) . \forall x : A . \pi_x \circ g = f_x : \mathrm{Prop}_0
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External meaning: "Every reindexing functor u^* of C has a right adjoint $\prod_u$ and the right adjoints satisfy the Beck-Chevalley condition."

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Proof.

Fix some lex base category $\mathcal{S}$. The standard proof of the special adjoint functor theorem is constructive and can be carried out in our language. In particular the proof is valid in the model $(\mathcal{S}, codis)$. Externalising the result gives us the fibered adjoint functor theorem for $\mathrm{Fib}_{\mathcal{S}}$. :) □

Separation Conditions

Def. Assume $(\mathcal{S}, W)$ is a site. An indexed category C is 0-,1-,2-separated when the functors to descent data are faithful, fully faithful, equivalences respectively.

$$C_\Gamma \longrightarrow Desc(C, \{u_i : \Delta_i \to \Gamma\}_i)$$

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Prop. (ext) is sound for C and all of its reindexings if and only if C is 0-separated.

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$$\frac{\Xi \vdash P : \exists! f : C_1(X,Y).\phi}{\Xi \vdash \iota f.\phi : C_1(X,Y)} \qquad \frac{\Xi \vdash \iota f.\phi : C_1(X,Y)}{\Xi \vdash * : \phi[\iota f.\phi / f]}$$

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Prop. When C is a prestack then definite description is sound for C.

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Meaning of 2-separatedness?

Prop. When C is a semicoflexible stack then it satisfies "1-definite choice" from the internal perspective.

Separation Conditions

Prop. When $\Gamma \vdash C : \mathrm{Cat}$ is a stack then..

$$\Gamma \Vdash \forall D : \mathrm{Cat}.\forall F : D^C.$$
$$\ulcorner F \textit{ is fully faithful and eso} \urcorner \to \ulcorner F \textit{ has an inverse} \urcorner$$

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Proof.

(classical) For each $d : D$ choose some $Gd : C$ together with an isomorphism $\alpha_d : FGd \to d$... □

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(classical) For each $d : D$ choose an initial object $\eta_d : d \to UFd$ in the comma category d/U.. □

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When C is a globally defined semicoflexible stack:

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$$\frac{\vdash C, D : \mathrm{Cat} \qquad \vdash F : D^C \qquad \Vdash \ulcorner F \textit{ pointwise has a right adjoint} \urcorner}{\vdash \mathbf{R}_F : C^D}$$

$$\frac{\textit{same assumptions}}{\vdash \varepsilon_F : (D^D)_1(F\mathbf{R}_F, 1_D)} \qquad \frac{\textit{same assumptions}}{\Vdash \ulcorner (\mathbf{R}_F, \varepsilon_F) \textit{ is a right adjoint of } F \urcorner}$$

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$$\frac{\Xi \vdash A, B : \mathrm{Set} \qquad \Xi \vdash p : \mathrm{Set}_1(\mathbf{1}, \mathbf{R}_\Delta(A, B))}{\Xi \vdash \varepsilon_\Delta(A, B)_1 \circ p : \mathrm{Set}(\mathbf{1}, A)}$$

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$$\frac{\Xi \vdash A, B : \mathrm{Set}}{\Xi \vdash A \times B : \mathrm{Set}}$$

$$\frac{\Xi \vdash a : A \qquad \Xi \vdash b : B}{\Xi \vdash (a, b) : A \times B}$$

$$\frac{\Xi \vdash A, B : \mathrm{Set} \qquad \Xi \vdash p : A \times B}{\Xi \vdash p_1 : A}$$

Set Constructors

Internal characterisation of set constructors:

$CwF(\mathcal{S})$ **admits**	**Internal characterisation**
strong Σ-types	"Set is infinitary extensive"
binary product types	"Set has binary products"
extensional identity types	"Set has equalizers of pairs $x, y : \mathbf{1} \rightrightarrows A$"
singelton type	"Set has a terminal object"
empty type	"Set has a strict initial object"
weak sum types	"Set has coproducts"
Π-types	"Set has small products"

Set Constructors

Internal characterisation of set constructors (when strong Σ-types and finite limits are already available):

$CwF(\mathcal{S})$ **admits**	**Internal characterisation**
bracket types	"Set has coequalizers of pairs $\pi_1, \pi_2 : A \times A \rightrightarrows A$ and the coequalizers are subsingelton sets"
effective quotient types	"Set has effective quotients of internal equivalence relations"
natural numbers	"Set has a natural number object"