From Type Theory to Homotopy Theory: Part I

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What I will present: context

When Voevodsky designed his model of the axiom of univalence, he started from the classical Quillen Model Structure on simplicial sets

E.g. the justification of dependent products

Using the fact that trivial cofibrations are stable under pullbacks along Kan fibrations (referred to as right proper in the literature) one easily establishes that Kan fibrations are closed under dependent products

From lecture notes by Thomas Streicher

What I will present: context

This is using highly non effective notions such as minimal fibrations

Types are interpreted as special (fibrant) simplicial sets, but the Quillen Model Structure is defined on all presheaves

What I will present: context

Type Theory is closely connected to constructive mathematics so it is natural to look for a constructive justification of the axiom of univalence

Such a constructive justification is provided by "cubical" set models, a class of presheaf models parametrised by an interval object and a "cofibration" classifier

Crucially, the interval needs to be *tiny* and this class of models does *not* cover the simplicial set model

This justification does not rely on building a Quillen Model Structure on all presheaves

What I will present: context

However, it has been noticed by Christian Sattler that one can “reverse” Voevodsky’s model construction, and define a Quillen Model Structure on all presheaves using these “direct” models of univalence!

One key component there is the fact that we have a *fibrant* universe of fibrant types

What I will present: context

Since one has a Quillen Model Structure on presheaves, one can wonder which of these models is Quillen equivalent to the classical Model Structure on spaces

For such model, one would then hope to get in this way a constructive explanation of *homotopy theory* using *type theory*

What I will present: context

Christian Sattler found out that, for several of these models (de Morgan, cartesian, BCH), the answer is *negative*

He also found out that an equivariant version of the cartesian model (j.w.w. Steve Awodey, Evan Cavallo, Emily Riehl and Christian Sattler) is such that the canonical realization functor *is* a Quillen equivalence

His argument however makes essential use of classical logic

Since then, Evan Cavallo and Christian Sattler found other models *classically* equivalent to spaces

What I will present: context

There is also a constructive candidate for the notion of homotopy types using semisimplicial sets, but it does not seem possible to refine this to a model of dependent type theory with univalence

This was the situation in 2019

Question

Mike Shulman *The Derivator of Setoids*, 2021

Can homotopy theory be developed in constructive mathematics, or even in ZF set theory without the axiom of choice?

Question

In particular, there are now at least two constructive homotopy theories - the aforementioned simplicial sets and the equivariant cartesian cubical sets of [ACC+21] - that can classically be shown to present the homotopy theory of spaces. However, it is not known whether they are constructively equivalent to each other. Thus one may naturally wonder: if they are not equivalent, which is the "correct" constructive homotopy theory of spaces? Or, perhaps, are they both "incorrect"? What does "correct" even mean?

What I will present

Recently, Christian Sattler found out that by refining one special model by a left exact modality (using some notions introduced for development of constructive sheaf models), one gets a *constructive notion of homotopy types* which should provide a "correct" notion of homotopy theory of spaces

This special model is the one based on presheaves over the category of finite nonempty posets $\square$

(Dually, this corresponds to finitely presented non degenerate distributive lattices, Birkhoff's Theorem)

This category contains the category Δ as a full subcategory

Plan of the talk

(1) General remarks on constructive presheaf models

Use of *internal language*, of *partial elements* and technique of *relativization*

Why this constructive approach can be interesting

Illustrate mix of ideas from homotopy and programming with presheaves

(2) How to build left exact modalities

(3) Δ_+, Δ and $\Box$

Dependent Type Theory

N.G. de Bruijn, system AUTOMATH, 1967

Constructive Presheaf Models

What is a model of type theory?

Generalised algebraic theory: sorts, operations, equations

This was done in the 80s, 90s: J. Cartmell, P.-L. Curien, Th. Ehrhard

Sorts: contexts, substitutions $\Theta \to \Gamma$, types $\mathsf{Type}(\Gamma)$, elements $\mathsf{Elem}(\Gamma, A)$

Extension operation: if $A : \mathsf{Type}(\Gamma)$ then $\Gamma.A$ is a context

$\mathsf{p} : \Gamma.A \to \Gamma$ $\qquad$ $\mathsf{q} : \mathsf{Elem}(\Gamma.A, A\mathsf{p})$

Constructive Presheaf Models

Operations: no canonical choices, but bi-interpretable theories

If $\sigma : \Theta \to \Gamma$ and A in $\mathsf{Type}(\Gamma)$ and u in $\mathsf{Elem}(\Theta, A\sigma)$ then $(\sigma, u) : \Theta \to \Gamma.A$

If $\sigma : \Theta \to \Gamma$ and A in $\mathsf{Type}(\Gamma)$ then $\sigma^+ : \Theta.A\sigma \to \Gamma.A$

If $u : \mathsf{Elem}(\Gamma, A)$ then $[u] : \Gamma \to \Gamma.A$

We have $\sigma^+ = (\sigma \mathsf{p}, \mathsf{q})$ and $[u] = (\mathsf{id}, u)$

Conversely we can define $(\sigma, u) = \sigma^+[u]$

Constructive Presheaf Models

Type defines a *presheaf* on the category of contexts (Dybjer, 95)

If A in Type(Γ) and $\sigma : \Theta \to \Gamma$ then $A\sigma$ in Type(Θ)

The extension operation $\Gamma.A$ can be connected to Grothendieck's notion of *representable* maps of presheaves (natural models, Awodey 2013)

Constructive Presheaf Models

To check that we have a model is to define the sorts and the operations and to check that the equations are satisfied

For the presheaf model, we assume to have a cumulative hierarchy of universes in the meta theory $\mathcal{U}_0,\ \mathcal{U}_1,\ldots,\mathcal{U}_\omega$ and a given (small) base category

The sort of *contexts* is interpreted by the *set* of $\mathcal{U}_\omega$-valued presheaves on this base category

If Γ is a context the sort $\mathsf{Type}(\Gamma)$ is interpreted by the *set* of $\mathcal{U}_\omega$-valued presheaves on the category of elements of Γ

If $A : \mathsf{Type}(\Gamma)$ then $\mathsf{Elem}(\Gamma, A)$ is interpreted by the set of global sections of the presheaf A

Constructive Presheaf Models: remarks

All equations have to be checked and to hold strictly

The metatheory does not need to be set theory

For instance, Mark Bickford was able to represent the notion of presheaf models in NuPrl, using extensional type theory as metatheory

Formalising Category Theory and Presheaf Models of Type Theory in NuPrl

and he formalised one cubical set model in NuPrl

See also the paper of Taichi Uemura

Cubical Assemblies and Independence of the Propositional Resizing Axiom

Constructive Presheaf Models: remarks

This is a natural generalization of the notion of Kripke model

A suggestive presentation for simply typed λ-calculus is in the paper of Dana Scott *Relating Theories of the λ-calculus*, 1980

For some models, the base category is a category of "names" and morphisms can be thought of as "renaming"

The required equations then don't need extensionality in the meta theory

The models are then quite similar to the models used for *higher-order abstract syntax* or *nominal* λ-calculus

Constructive Presheaf Models: relativization

We get in this way a model of dependent type theory with a cumulative hierarchy of universes V_n

We will use the convenient notations of dependent type theory to describe what is going on in such models

Constructive Presheaf Models: relativization

We don't get yet however an interpretation of type theory with equality and univalence

In order to get such a model, we use the fundamental and simple technique of *relativization*

We define an operation $\mathsf{Fib} : \mathsf{V}_n \to \mathsf{V}_n$ such that Fib is closed by type theoretic operations

$\Pi_{A:\mathsf{V}_n}\Pi_{B:A\to\mathsf{V}_n}\mathsf{Fib}(A) \to (\Pi_{a:A}\mathsf{Fib}(B\ a)) \to \mathsf{Fib}(\Pi\ A\ B)$

$\Pi_{A:\mathsf{V}_n}\Pi_{B:A\to\mathsf{V}_n}\mathsf{Fib}(A) \to (\Pi_{a:A}\mathsf{Fib}(B\ a)) \to \mathsf{Fib}(\Sigma\ A\ B)$

$\mathsf{Fib}(\Sigma_{X:\mathsf{V}_n}\mathsf{Fib}(X))$

Constructive Presheaf Models: relativization

We get a new model by redefining $\mathsf{Type}(\Gamma)$ as being the set of types A *together* with a proof/element c_A of the type $\mathsf{Fib}(A)$

$$\mathsf{Type}_c(\Gamma) = \sum_{A:\mathsf{Type}(\Gamma)} \mathsf{Fib}(A)$$

An element of A, c_A is an element of A

$$\mathsf{Elem}_c(\Gamma, (A, c_A)) = \mathsf{Elem}(\Gamma, A)$$

Constructive Presheaf Models: relativization

We can define $\mathsf{U}_n = \Sigma_{X:\mathsf{V}_n}\mathsf{Fib}(X)$

This is a type theoretic formulation of the *classifying type for fibration structures* presented in Steve's lecture

How to define Fib?

Internal language

Presheaf models contain a "new" kind of elements: partial elements

$A_p = \Sigma_{\psi:\Omega} A^{T(\psi)}$, where $T(\psi)$ subsingleton $\{0 \mid \psi\}$

We work with a subpresheaf Φ of Ω and $A_p = \Sigma_{\psi:\Phi} A^{T(\psi)}$

We can define when a *total* element $a : A$ extends a *partial* element ψ, u by the statement $\forall_{x:T(\psi)} a = u\ x$

This forms the "extension" type $A[\psi, u]$, which is a subtype of A

We can define what it means for $A : \mathsf{V}_n$ to be *contractible*

$\mathsf{isContr} : \mathsf{V}_n \to \mathsf{V}_n \quad \mathsf{isContr}\ A = \Pi_{(\psi,u):A_p} A[\psi, u]$

Internal language

For defining the notion of *fibrant* family of types, we need to assume to have an “interval”, a type $\mathbf{I}$ with 0 and 1 and $0 \neq 1$

This notion of fibrant family can then be written as a type theoretic notion, which corresponds to a general Lemma that was extracted by Eilenberg (1939), and which I conjecture was an inspiration for Kan’s notion of fibration

If A *subpolyhedra of* B *and given two homotopic functions* $f_0, f_1 : A \to X$ *and an extension* $f_0' : B \to X$ *then there is an extension* f_1' *of* f_1 *homotopic to* f_0'

Proofs of basic results about homotopy *can be obtained quite neatly by repeated, and sometimes tricky, use of this general lemma* (Eilenberg, 1939)

Internal language

We represent the notion of “subpolyhedra” by the notion of subpresheaf classified by Φ

We can represent internally the notion of fibrant *family* of types

$\mathsf{comp} : \Pi_{A:\mathsf{V}_n}(A \to \mathsf{V}_n) \to \mathsf{V}_n$

$\mathsf{fill} : \Pi_{A:\mathsf{V}_n}(A \to \mathsf{V}_n) \to \mathsf{V}_n$

Internal language

comp(A, B) type of operations taking an element $\gamma : A^{\mathbf{I}}$ and $\psi : \Phi$ and $v : \Pi_{x:\mathbf{I}} T(x = 0 \vee \psi) \to B(\gamma x)$ and producing an element in $B(\gamma 1)[p, v\ 1]$

fill(A, B) type of operations taking an element $\gamma : A^{\mathbf{I}}$ and $\psi : \Phi$ and $v : \Pi_{x:\mathbf{I}} T(x = 0 \vee \psi) \to B(\gamma x)$ and producing an element in $\Pi_{x:\mathbf{I}} B(\gamma x)[x = 0 \vee \psi, v\ x]$

(Together with dual operations swapping 0 and 1)

This can be seen as a refinement of *transport* operations: in particular we have maps $B\ 0 \to B\ 1$ by comp and *path lifting* by fill for any given path γ in the base in the case where ψ is $\bot$

Internal language

If $\mathbf{I}$ has a lattice structure, then it is direct to build an element of $\mathsf{comp}(A,B) \rightarrow \mathsf{fill}(A,B)$

$f_B\ \psi\ \gamma\ v\ x = c_B\ \psi\ \gamma_x\ w_x$

with

$\gamma_x\ y = \gamma\ (x \wedge y)$ and

$w_x : \Pi_{y:\mathbf{I}} T(\psi \vee x = 0 \vee y = 0) \rightarrow B(\gamma(x \wedge y))$

$w_x\ y = w\ (x \wedge y)$

Internal language

Crucially in this argument we only need the equalities

$$x \wedge 0 = 0 \qquad 0 \wedge y = 0$$

but we don't need $(x \wedge y) = 0$ to be $x = 0 \vee y = 0$

A detailed analysis of the required properties of $\mathbf{I}$ and Φ and a formulation in Agda can be found in the work(s) of Ian Orton and Andy Pitts

Internal language

On the other hand, it is also direct to build an element of

$\mathsf{fill}(A,B) \to \mathsf{comp}(\Sigma_A B, D) \to \mathsf{comp}(A, \lambda_{a:A}\Pi_{b:Ba}D(a,b))$

$c_{\Pi_B D}\ \psi\ \gamma\ l\ b_1 = c_D\ \psi\ (\gamma, b)\ (\lambda_{x:\mathbf{I}} l\ x\ (bx))$

where b is the path lifting of b_1 with $b : \Pi_{x:\mathbf{I}} B(\gamma x)$ satisfying $b1 = b_1$

This provides a simple and effective proof that fibrant families are closed by dependent products

This argument applies to the case of simplicial sets and should be compared with the usual argument, which needs to have first built a *right proper* Quillen Model Structure

Internal language: two remarks

Note that for getting the notion of fibrant family of types (corresponding to the notion of Kan fibration), we only need to consider the case where A and B are *globally* defined

But in this internal approach we have $\mathsf{fill} : \Pi_{A:\mathsf{V}_n}(A \to \mathsf{V}_n) \to \mathsf{V}_n$ and $\mathsf{fill}(A,B)$ is defined even for locally defined families of types

We have a natural decomposition $\mathsf{comp}(A,B) = \Pi_{\gamma:A^{\mathbf{I}}}\mathsf{Comp}(B \circ \gamma)$ with $\mathsf{Comp} : \mathsf{V}_n^{\mathbf{I}} \to \mathsf{V}_n$

$\mathsf{Comp}(P) = \Pi_{\psi:\Phi}\Pi_{u:\Pi_{x:\mathbf{I}}T(\psi \vee x=0)\to P\ x}\ P\ 1[\psi, u\ 1]$

Internal language

So far, all that we have presented works in particular for *simplicial* sets, presheaf models over Δ taking Φ to be Ω and the interval $\mathbf{I}$ to be Δ^1

The notion of Kan fibration corresponds then to the characterisation found first I believe by Gabriel-Zisman 67

This is *classically* equivalent to Kan's definition in term of Horn filling

Modulo this equivalence, this provides a simple alternative proof that Kan fibrations are closed by dependent products and hence that the associated Quillen Model Structure is right proper

Tiny interval

In order to build the universe of *fibrant* types constructively we need the interval $\mathbf{I}$ to be *tiny* (as explained in the previous talk by Steve)

We can then define $\mathsf{Fib} : \mathsf{V}_n \to \mathsf{V}_n$ such that

(1) *internally*, we have an operation

$$\Pi_{A:\mathsf{V}_n}\Pi_{B:A\to\mathsf{V}_n}(\Pi_{a:A}\mathsf{Fib}(B\ a)) \to \mathsf{comp}(A, B)$$

(2) for A, B *globally* defined, we have an operation $\mathsf{comp}(A, B) \to \Pi_{a:A}\mathsf{Fib}(B\ a)$ but this operation *cannot* be defined internally

Interpretation of type theory

We can then define the universe of fibrant types to be $\mathsf{U}_n = \Sigma_{X:\mathsf{V}_n}\mathsf{Fib}(X)$

We can build elements of type

$\Pi_{A:\mathsf{V}_n}\Pi_{B:A\to\mathsf{V}_n}\mathsf{Fib}(A) \to (\Pi_{a:A}\mathsf{Fib}(B\ a)) \to \mathsf{Fib}(\Pi\ A\ B)$

$\Pi_{A:\mathsf{V}_n}\Pi_{B:A\to\mathsf{V}_n}\mathsf{Fib}(A) \to (\Pi_{a:A}\mathsf{Fib}(B\ a)) \to \mathsf{Fib}(\Sigma\ A\ B)$

$\mathsf{Fib}(\Sigma_{X:\mathsf{V}_n}\mathsf{Fib}(X))$

Interpretation of type theory

Hence by taking as new notion of types A *together* with an element of Fib(A) we get a new model of type theory

We can define a notion of *path* types $\mathsf{Path}_A\ a_0\ a_1 = \Pi_{x:\mathbf{I}} A[x = 0 \vee x = 1, v]$ where $v\ z = a_0$ for $z : T(x = 0)$ and $v\ z = a_1$ for $z : T(x = 1)$

If we have B family of types over A and $\Pi_{a:A}\mathsf{Fib}(B\ a)$ then we have $\mathsf{comp}(A, B)$ and in particular we have transport

We can then interpret type theory with univalence, *except* that we only interpret the computation rule of identity type in a *propositional* and not *judgemental* way

Interpretation of type theory

Andrew Swan found out, using ideas from Quillen Model Structure, how to interpret the *judgemental* computation rule of identity type

An element of identity type is a pair ψ, ω where ω is a path which is strictly constant on ψ

This is an unexpected application of ideas from Quillen Model Structure!

Evan Cavallo and Bob Harper found then that we can use ideas from interpretation of Higher Inductive Types (explained later) to give another interpretation of identity types

These two examples illustrate well the possible combinations of ideas coming from functional programming with presheaves and coming from homotopy theory

Internal language: benefits

(1) All this has been formalised in NuPrl and in Agda

(2) The proof that fibrant types are preserved by dependent products is simple and effective

(3) The same remark that the filling operation can be reduced to the composition operation suggests a simple and effective proof that the universe of fibrant types is itself fibrant

(4) The composition operation can be naturally decomposed into *homogeneous composition* and *transport*; this provides a semantics to *parametrised* Higher Inductive Types and the same idea can be applied to the simplicial set model (it was not known before this how to interpret pushouts or suspension in this model)

Internal language: benefits

Homogeneous composition for a type A is an operation hcomp ψ u which takes as arguments $\psi : \Phi$ and $u : \Pi_{x:\mathbf{I}} A^{T(x=0 \vee \psi)}$ and produces an element in $A[\psi, u\ 1]$

The corresponding filling produces an element in $\Pi_{x:\mathbf{I}} A[x = 0 \vee \psi, u\ x]$

This expresses exactly that $\Sigma_A \mathsf{Path}_A\ a$ is contractible for all $a : A$

Any partial element z of $\Sigma_A \mathsf{Path}_A\ a$ is such that $z.1$ has a total extension

For A *global* this expresses that A is fibrant

Internal language: benefits

Let A be any type

If we have a relation R on A such that $\mathsf{Path}\ a\ b \to R\ a\ b$ and $\Sigma_A R\ a$ is *contractible* then we get a homogenenous composition for A:

any partial element z of $\Sigma_A \mathsf{Path}\ a$ is such that $z.1$ has a total extension

In this way we get that U_n is *fibrant* from *univalence*

Indeed univalence can be formulated as the fact that $\Sigma_{\mathsf{U}_n} \mathsf{Equiv}\ X$ is contractible for all $X : \mathsf{U}_n$ and we have $\mathsf{Path}\ X\ Y \to \mathsf{Equiv}\ X\ Y$

Internal language: benefits

This argument can also be used for the simplicial set model

The original proof by Voevodsky that the universe is fibrant uses the notion of minimal fibration

Internal language: benefits

Connected to (4), it is possible to define the *fibrant* replacement A_f of a *global* type A by a “new” kind of inductive definition, which involves the notion of partial element

We have constructors

$\mathsf{inc} : A \to A_f$ and

$\mathsf{hcomp}\ \psi\ u\ x : A_f[\psi \vee x = 0, u\ x]$

for $\psi : \Phi$ and $u : \Pi_{x:\mathbf{I}} A_f^{T(\psi \vee x = 0)}$ and $x : \mathbf{I}$

Internal language: benefits

This is one key component for decomposing any map into a trivial cofibration and a fibration

$A \mapsto A_f$ is functorial and we can define $\alpha : A \to B$ to be an equivalence if $\alpha_f : A_f \to B_f$ is a homotopy equivalence

All this has been formalised in the PhD thesis of Simon Boulier 2018

Constructive models: don't be afraid of universes

(1) Dependent type theory with universes is much weaker than ZFC with universes (technically, even much weaker than Π^1_2-comprehension, cf. Martin-Löf's paper on *The Hilbert-Brouwer controversy resolved?*)

In particular, we get the result that *the axiom of univalence does not add any proof theoretic power to type theory*

(2) Direct to have presheaf models and then sheaf models

Joyal had to use the technique of "Boolean localisation" (Barr's Theorem) in his 1984 letter to Grothendieck in order to define a Quillen Model Structure on simplicial sheaves

Constructive models: benefits

(3) "Direct" to have recursive models, cf. work of Taichi Uemura and Andrew Swan

Main techniques

Use of *internal language*, of *partial elements* and technique of *relativization*

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