From Type Theory to Homotopy Theory: Part II

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An example of a model

$\square$ category of finite nonempty posets

The objects are denoted by $X, Y, \ldots$

This category contains the object $[n]$, linear poset with $n+1$ elements

We take the interval $\mathbf{I}$ to be $Yo([1])$

Since $\square$ has finite product, $\mathbf{I}$ is tiny

(Over Δ, the interval $Yo([1])$ is not tiny)

An example of a model

Any subset S of X defines a sieve on X: sieve of maps having image inside S

We define $\Phi(X)$ to be the set of sieves that are finite sup of such sieves

An element of $\Phi(X)$ is determined by a finite collection $S_1, \dots, S_n$ and sieve of maps having image inside one S_i

We can have $n = 0$ which corresponds to the empty sieve

Then Φ is a strict subpresheaf of Ω: the sieve determined by $\{0, 1\} \to [1]$ is not in $\Phi([1])$

Quillen Model Structures

The associated Quillen Model Structure satisfies the following properties

-*Frobenius* (and right properness: trivial cofibrations and equivalences are preserved by pullbacks along fibrations)

-*Equivalence Extension Property*

-*Fibration Extension Property* (a.k.a. “Joyal’s trick”)

Example (due to Christian Sattler)

Let $A = Yo(\{0,1\})$ then the diagonal map $A \to A^{\mathbf{I}}$ is an isomorphism

Hence A is such that any family of types B over A is fibrant

A has two global points $1 \to A$ and we have the two maps $\delta_0, \delta_1 : A \to \Phi$

In particular $B = \lambda_{a:A} T(\delta_0 a) + T(\delta_1 a)$ is fibrant and is a family of (strict) propositions

($T(\psi)$ is the subsingleton corresponding to ψ)

For any global points $a : 1 \to A$ we have that Ba is the unit type

But B has no global section

Example (due to Christian Sattler)

We have an example of a fibration $B \to A$ (family of propositions)

Any pullback along $1 \to A$ is contractible

$B \to A$ has no section

It follows that the Quillen Model Structure on presheaves over $\square$ cannot be equivalent to spaces!

We are going to build a suitable relativization of this model which will be equivalent to spaces

Question

Mike Shulman *The Derivator of Setoids*, 2021

Can homotopy theory be developed in constructive mathematics, or even in ZF set theory without the axiom of choice?

Question

In particular, there are now at least two constructive homotopy theories - the aforementioned simplicial sets and the equivariant cartesian cubical sets of [ACC+21] - that can classically be shown to present the homotopy theory of spaces. However, it is not known whether they are constructively equivalent to each other. Thus one may naturally wonder: if they are not equivalent, which is the "correct" constructive homotopy theory of spaces? Or, perhaps, are they both "incorrect"? What does "correct" even mean?

Some troubling points

While it is difficult to formulate what does "correct" even mean, one can list some trouble points that have appeared while working with these models from a constructive point of view

Some troubling points

(1) Like for the groupoid model, one would expect countable choice to hold in these models, but this does not seem to be the case

(2) Related to the last point, one would expect that propositional truncation can be defined like in the setoid model/Bishop notion of set, but this is not the case

(3) Some of these models validates the *negation* of excluded-middle!

(4) Too many models!

Some troubling points

It is possible to define a modified form of Model Structure on simplicial sets *with* decidable degeneracies (work of Simon Henry) and this is related to a previous attempt of defining a semisimplicial model of type theory (work with Bruno Barras and Simon Huber)

However, we don't seem to get a model of dependent type theory in this way

We can interpret a weak form of dependent products, and it does not seem possible to "strictify" the model

A positive point

There has been however some positive points, e.g. the models are developped in weak metatheory

One most important positive point may be the following

It is direct to define presheaf models of dependent type theory with univalence

If we have a model over a base category $\Box$ and we want to define a presheaf model over a category C we reproduce the construction of the model with the base category $C \times \Box$

A positive point

Such a simple construction of general presheaf models cannot be done if one justifies univalence using classical logic

A positive point

It is then possible to define *sheaf* models by *relativization*

This is the technique of forcing: we force a family of propositions $p_r,\ r : X$ to be true by relativizing the model to types A such that all diagonal maps $A \to A^{p_r}$ are equivalence

This defines a *property* $\mathsf{C} : \mathsf{U}_n \to \mathsf{U}_n$ of A

A positive point

We have

$\Pi_{A:\mathsf{U}_n}\Pi_{B:A\to\mathsf{U}_n}(\Pi_{a:A}\mathsf{C}(B\ a)) \to \mathsf{C}(\Pi\ A\ B)$

$\Pi_{A:\mathsf{U}_n}\Pi_{B:A\to\mathsf{U}_n}\mathsf{C}(A) \to (\Pi_{a:A}\mathsf{C}(B\ a)) \to \mathsf{C}(\Sigma\ A\ B)$

$\mathsf{C}(\Sigma_{X:\mathsf{U}_n}\mathsf{C}(X))$

Hence we get a new model of univalence by relativization

Left exact modalities

This also works for data types and higher inductive types

We add new constructors, e.g. N will also have constructors

$0 : N$

$\mathsf{S} : N$

$\mathsf{patch} : \Pi_{r:X} N^{p_r} \to N$

$\mathsf{linv} : \Pi_{n:N} \mathsf{Path}_N\ n\ (\mathsf{patch}\ (\delta\ n))$ with $\delta\ n\ z = n$

This forces N to be such that $\delta : N \to N^{p_r}$ is an equivalence

Left exact modalities

This has been used to get, in a constructive metatheory, a model of Synthetic Algebraic Geometry

The base category is the category of f.p. k-algebra for some fixed commutative ring k

One has then the generic k-algebra R and one *forces* this ring to be local using the Zariski topology

Here the family of propositions is $inv(r_1) \vee \cdots \vee inv(r_n)$ for any $r_1, \ldots, r_n$ such that $1 = (r_1, \ldots, r_n)$ in R

A positive point

What is crucial is that $A \to A^p$ defines a *strict left exact modality* if p is a proposition

There are however *new* kinds of strict left exact modalites that do not come from propositions

Descent data

In particular the notion of descent data can be seen to define a strict left exact modality (closely connected to the cobar operation used by Mike Shulman in his semantics in higher topos, but, in this setting, it defines a *left exact modality*)

$DF(X)$ is the collection of family of points $u(f_0)$ in $F(X_0)$ for $f_0 : X_0 \to X$ and family of lines $u(f_0, f_1) : u(f_0)f_1 \to u(f_0 f_1)$ for $f_1 : X_1 \to X_0$ and family of triangles $u(f_0, f_1, f_2)$ for $f_2 : X_2 \to X_1$ and so on

One can show that this defines a *strict* left exact modality on any given presheaf model

Lex operation

We axiomatised what is going on in previous work with F. Ruch and Ch. Sattler by the notion of *lex operation*

This is a *strict* pointed functor D, with a natural transformation $\eta_A : A \to DA$. acting also on families; we extracted a necessary and sufficient condition for this to define a left exact modality

η_{DA} *and* $D\eta_A : DA \to D^2A$ *should be path equal and should be equivalence*

By localisation, we get a (strict) model of dependent type theory with univalence and higher inductive types

Lex operation

A typical lex operation is exponentiation $DA = A^R$ for a fixed type R

We have $D(\Sigma_{x:A} Bx)$ isomorphic to $\Sigma_{u:DA}\Pi_{r:R} B(ur)$ and we also have an action on families

$D_f B : DA \to \mathsf{U}$ if $B : A \to \mathsf{U}$

with $(D_f B)u = \Pi_{r:R} B(ur)$

This lex operation is a left exact modality if R is a *proposition*

We have $\eta_A : A \to A^R$

And in this case, η_{DA} and $D\eta_A$ are path equal and are equivalences

Descent data

If we relativize w.r.t. this descent data modality we get a new model of type theory

We can look at the associated Quillen Model Structure

This becomes the *injective* model structure: a weak equivalence $\alpha : A \to B$ is equivalence a map such that each maps $\alpha_X : A(X) \to B(X)$ are equivalences

This has been used by D. Licata and M. Weaver to get constructive models of directed type theory

A new model

I will now describe a new insight (due to Christian Sattler) which seems to provide a positive answer to Mike Shulman's question

Can homotopy theory be developed in constructive mathematics, or even in ZF set theory without the axiom of choice?

This uses the technique of relativization w.r.t. a left exact modality which is not propositional

Δ and $\Box$

(All I present from now on is due to Christian Sattler)

Δ is the category of finite nonempty linear posets

Δ_+ is the category of finite nonempty linear posets, with injective maps

$\Box$ is the category of finite nonempty posets

This is the dual of the category of nondegenerate f.p. distributive lattices and it corresponds to a version of cubical type theory having nice properties

$$\Delta_+ \to \Delta \to \Box$$

Δ_+ and $\Box$

We have the inclusion $\Delta_+ \to \Box$

This defines a strict monad D on the presheaf model $Ps(\Box)$

It is direct to check that this strict monad satisfies that η_{DA} and $D\eta_A$ are path equal and are equivalences (we don't even need A to be fibrant to build such a path)

What is a little more subtle is that DA is fibrant if A is fibrant

Hence D is a *strict left exact modality*

By localisation a model of dependent type theory with univalence and higher inductive types

Δ_+ and $\square$

The objects $X, Y, \ldots$ of $\square$ are finite nonempty posets

In particular we have the linear poset $[n]$ with $n+1$ elements

Δ is a full subcategory, and Δ_+ a subcategory

If A in $\mathsf{Type}(\Gamma)$ and ρ in $\Gamma(X)$ then an element u of $DA(X,\rho)$ is a family $u(f) : A([n], \rho f)$ for $f : [n] \to X$ such that $u(fg) = u(f)g$ whenever $g : [m] \to [n]$ is stricly monotone

This allows arguments by induction on dimension, which is a key component of the semisimplical set model

Δ_+ and $\Box$

This relativized model does not have any of the troubling points we listed above

(1) countable choice holds (even if it does not hold in the meta theory)

(2) propositional truncation can be defined as expected

(3) Whitehead principle holds: a map $f : X \to Y$ is an equivalence iff $\pi_0(f)$ bijection and all $\pi_n(f, x)$ are isomorphisms

Δ_+ and $\square$

(1) follows from the key property

Let A be a type over Γ which is a proposition, with A which is D-modal, then

any section on points can be extended to a global section

Consequences for Cubical type theory

Syntactically, we work with contexts that contain a *finite presentation* of a non degenerate *distributive lattice*, and we still have normalisation and decidable type checking

All the syntactical operations (as long as we do not use reversal) are validated by this semantics

Classical compatibility with excluded-middle and axiom of choice

Δ and $\Box$

Christian Sattler also noticed that if we don't relativize the presheaf model $Ps(\Box)$ this model is not equivalent to spaces

Over the (representable) context $\Gamma = x : \mathbf{I},\ y : \mathbf{I},\ x \wedge y = 0,\ x \vee y = 1$ we have the family of strict proposition $A = (x = 1) + (y = 1)$

For any global point $\rho : 1 \to \Gamma$, the type $A\rho$ is contractible

But A is not contractible $\mathsf{Elem}(\Gamma, A)$ is empty

This shows that $Ps(\Box)$ satisfies the *negation* of excluded-middle

(Note that DA is contractible as it should be)

Δ and $\Box$

We obtain a Quillen Model Structure on $Ps(\Box)$, which not only corresponds to spaces classically, but should also be well behaved constructively

This should provide a constructive explanation of homotopy types!

For instance, we can define the nerve of a category by taking $N(C)(X)$ to be the set of functors from X to C and we expect Quillen's Theorems A and B to be (constructively) valid in this setting

References

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