HOL Light from the foundations

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The world of interactive theorem provers

A few notable general-purpose theorem provers

There is a (too?) diverse world of interactive theorem provers, e.g

- ACL2
- Agda
- Coq
- HOL (HOL Light, HOL4, ProofPower, HOL Zero, Candle)
- IMPS
- Isabelle
- Lean
- Metamath
- Mizar
- Nuprl
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See Wiedijk's *The Seventeen Provers of the World* for descriptions of many systems and proofs that $\sqrt{2}$ is irrational.

Different traditions, different characteristics

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- Implementation and software architecture
- Proof language
- Level of automation
- Pre-proved library size and content

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See Harrison, Urban and Wiedijk *History of Interactive Theorem Proving* for more on early ITP history and origins.

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- Partly as a result of their computer science interconnections, many provers are based on type theory
  - Simple type theory (HOL family, Candle, Isabelle/HOL)
  - Martin-Löf type theory (Agda, Nuprl)
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  - Other typed formalisms (IMPS, PVS, HOL Light QE)
  - Univalent foundations (HoTT, Unimath, rzk, ...)
- Some are even based on very simple foundations analogous to primitive recursive arithmetic, without explicit quantifiers (ACL2, NQTHM)

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The checker or kernel can be much simpler than the prover as a whole, even to the point where it can be verified (Milawa, Candle).

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Mizar pioneered the declarative style of proof, but key ideas adopted elsewhere, e.g. Isabelle's structured proof language Isar.

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- Reimplement algorithms to perform proofs as they proceed
- Have suitable 'certificates' produced by an external tool checked in the inference kernel.
- Extend kernel with verified implementation (*reflection*).

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- The earliest large mathematical library is the Mizar Mathematical Library (MML), following the style of mathematical papers with extracted text and references.
- Many theorem provers including Coq, HOL Light, Isabelle/HOL and Lean have large and every-expanding libraries.

HOL Light and the LCF approach

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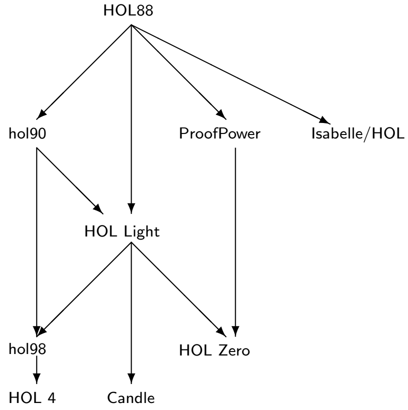
- HOL Light is designed to have a particularly simple and clean logical foundation.
- Written in Objective CAML (OCaml), a somewhat popular variant of the ML family of languages.
- Has been used for proofs in both verification and mathematics, including the Kepler conjecture (Flyspeck project)

The HOL family DAG

There are many HOL provers, all descended from Mike Gordon's original HOL system in the late 1980s.



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- LCF gives a very attractive mix of *security* and *extensibility/programmability*.
- There have been quite a few LCF-style provers for various logics, e.g. HOL, Nuprl, LAMBDA, Isabelle/HOL (and to some extent Coq used the LCF approach).

How an LCF-style prover works

A logical inference rule such as $\Rightarrow$-elimination (*modus ponens*)

$$\frac{\Gamma \vdash p \Rightarrow q \qquad \Delta \vdash p}{\Gamma \cup \Delta \vdash q}$$

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- An abstract type of *theorems* can restrict the user to an approved selection of *primitive inference rules* — all theorems must be created with those.
- By layers of programming, much more high-level and convenient *derived inference rules* can be programmed on top.

HOL Light

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- 10 rather simple primitive inference rules
- 2 conservative definitional extension principles
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Arguably, HOL Light is the computer-age descendant of Whitehead and Russell's *Principia Mathematica*:

- The logical basis is simple type theory, which was distilled (Ramsey, Chwistek, Church) from PM's original logic.
- Everything, even arithmetic on numbers, is done from first principles by reduction to the primitive logical basis.

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- HOL Light uses camlp5 to make a few modifications to OCaml's usual concrete syntax, which makes things slightly more complicated.
- There are also many similarities between OCaml (the 'metalogic') and the higher-order logic of HOL (the 'object logic'), which can be both illuminating and confusing.

Candle: a verified HOL Light

This project also gives an alternative fully verified version:

`https://cakeml.org/candle`

see Abrahamsson, Myreen, Kumar and Sewell, *Candle: A Verified Implementation of HOL Light*, ITP 2022.

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This is based on a completely different software stack, CakeML, a formally verified (in HOL4) ML compiler.

HOL Light applications

Tom Hales and The Kepler conjecture

The Kepler conjecture

The *Kepler conjecture* states that no arrangement of identical balls in ordinary 3-dimensional space has a higher packing density than the obvious 'cannonball' arrangement.

Hales, working with Ferguson, arrived at a proof in 1998:

- 300 pages of mathematics: geometry, measure, graph theory and related combinatorics, ...
- 40,000 lines of supporting computer code: graph enumeration, nonlinear optimization and linear programming.

Hales submitted his proof to *Annals of Mathematics* ...

The response of the reviewers

After a full four years of deliberation, the reviewers returned:

> *"The news from the referees is bad, from my perspective. They have not been able to certify the correctness of the proof, and will not be able to certify it in the future, because they have run out of energy to devote to the problem. This is not what I had hoped for.*
> *Fejes Toth thinks that this situation will occur more and more often in mathematics. He says it is similar to the situation in experimental science — other scientists acting as referees can't certify the correctness of an experiment, they can only subject the paper to consistency checks. He thinks that the mathematical community will have to get used to this state of affairs."*

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Dissatisfied with the state of affairs, Hales initiated a project called *Flyspeck* ('Formal Proof of the Kepler Conjecture') to completely formalize the proof.

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- Nonlinear optimization parts have been verified in using highly optimized derived rules for interval reasoning in HOL Light (Alexey Solovyev).
- The graph enumeration process has been verified in Isabelle/HOL via ML code generation (Tobias Nipkow).

Flyspeck and the future of mathematics

"In truth, my motivations for the project are far more complex than a simple hope of removing residual doubt from the minds of few referees. Indeed, I see formal methods as fundamental to the long-term growth of mathematics. (Hales, The Kepler Conjecture*)*

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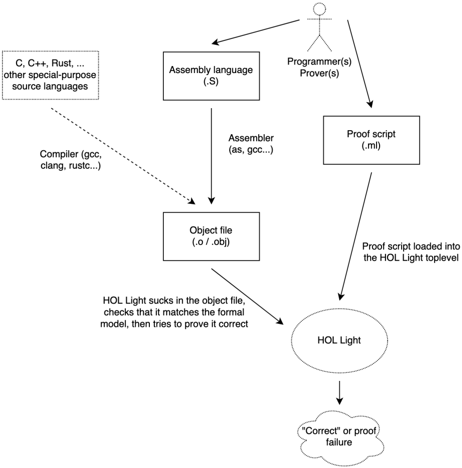
`https://github.com/awslabs/s2n-bignum`

Coding and verification flow



Verifying the actual code

Formalization of code as byte sequence derived from the object file:

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define_assert_from_elf "bignum_montmul_p256_mc"
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Run in continuous integration on any github pull request

Installation and OCaml basics

Installing OCaml

There are standard packages for OCaml, but it can be fiddly getting an appropriate version of OCaml and camlp5, so we recommend Opam:

`https://opam.ocaml.org/doc/Install.html`

Then this should give suitable prerequisites for HOL Light:

```
opam init
eval `opam env`
opam switch create 4.14.0
eval `opam env`
opam pin add camlp5 8.00.04
opam install num camlp5 camlp-streams ocamlfind
```

Installing HOL Light

Clone the git repo:

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git clone https://github.com/jrh13/hol-light.git
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Start OCaml and load the HOL Light root file

```
ocaml
#use "hol.ml";;
```

Wait 1-2 minutes for everything to load

The OCaml toplevel

When using HOL Light, you are in the top-level read-eval-print loop of OCaml, a strongly typed functional programming language.

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- Enter phrases terminated by *double* semicolon ';;' for evaluation

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and OCaml responds with

```
val it : int = 4
#
```

It not only returns the *value* (4) but also infers the type (int) and binds it to a name (it).

OCaml bindings

We can now use the name ‘it’ to stand for that expression:

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We can also choose our own names for bindings using ‘let *name* $=$ *expression*’, with multiple parallel bindings separated by ‘and’:

```
# let a = 2 and b = 3;;
val a : int = 2
val b : int = 3
# let c = a - b;;
val c : int = -1
```

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val it : int = 16
```

We can also choose our own names for bindings using 'let *name* = *expression*', with multiple parallel bindings separated by 'and':

```
# let a = 2 and b = 3;;
val a : int = 2
val b : int = 3
# let c = a - b;;
val c : int = -1
```

or make bindings *local* to an expression using 'in':

```
# let d = a / 2 in d + 6;;
val it : int = 7
# d;;
Error: Unbound value d
```

Basic OCaml datatypes

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- Booleans (bool), with elements `false` and `true` and operations like infix '`&&`' and '`||`'
- Strings (string) written in `"Double quotes"` with '`^`' as infix concatenation.

Pairs and lists

OCaml has two especially important structured datatypes, though the user can define more (and HOL Light defines its own for logical concepts);

- Pairs, written with an infix ‘,’ (the parentheses are only needed to establish precedence)

  ```
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  ```

Structured types can be nested in arbitrary ways (lists of pairs of lists etc.) and OCaml automatically keeps track of the types.

OCaml functions

One can define *functions* in OCaml using either of the following more or less equivalent forms:

- An explicit 'lambda' written '`fun v -> e`', e.g.

```
# let square = fun x -> x * x;;
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  # let square x = x * x;;
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  ```

Functions are applied just by juxtaposition; parentheses are only needed to establish precedence

```
# square 12 + 1;;
val it : int = 145
# square (12 + 1);;
val it : int = 169
```

Recursion and pattern-matching

Function definitions can be recursive with the `rec` keyword, and since OCaml is primarily a functional language, this is a major control flow mechanism.

- The factorial function can be defined as

```
# let rec fact n = if n <= 0 then 1 else n * fact(n - 1);;
val fact : int -> int = <fun>
# fact 12;;
val it : int = 479001600
```

- The length of a list can be determined as follows; note the use of pattern-matching 'match ... with' clauses:

```
# let rec length l =
  match l with
    [] -> 0
  | h::t -> 1 + length t;;
val length : 'a list -> int = <fun>
# length [1;2;3];;
val it : int = 3
```

Currying

OCaml allows function types to be nested, so one can implement multiple-argument functions as functions returning functions ('currying').

```
# let add x y = x + y;;
val add : int -> int -> int = <fun>
# let suc = add 1;;
val suc : int -> int = <fun>
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# suc 2;;
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```

Alternatively one can explicitly use a paired argument:

```
# let add(x,y) = x + y;;
val add : int * int -> int = <fun>
# add(1,3);;
val it : int = 4
```

Polymorphism

OCaml infers 'most general' types for functions according to an elegant polymorphic type system, with 'type variables' used to signify generality.

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# let identity x = x;;
val identity : 'a -> 'a = <fun>
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OCaml infers 'most general' types for functions according to an elegant polymorphic type system, with 'type variables' used to signify generality.

```
# let identity x = x;;
val identity : 'a -> 'a = <fun>
```

Such a function can be applied to any specific instance (or a more complex polymorphic type)

```
# identity 1;;
val it : int = 1
# identity false;;
val it : bool = false
```

HOL Light basics

Basic logical entities in OCaml

There are three key OCaml datatypes used to represent logical entities in HOL:

- Higher-order logic *types*, `hol_type`. You can conveniently create them using specially parsed backquotes with colon:

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# `:bool`;;
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- HOL terms, `term`, which can also be conveniently created via special parsing support

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# `1 + 2`;;
val it : term = `1 + 2`
```

- HOL theorems, which cannot be just created arbitrarily but must be *proved*, e.g. the pre-existing theorem that addition is commutative.

```
# ADD_SYM;;
val it : thm = |- !m n. m + n = n + m
```

Abstract type encapsulation

All the three core logical datatypes are effectively abstract data types, so how you can form them is *restricted* to ensure logical coherence

- You can only create HOL types that have been declared

```
# `:int triple`;;
Exception: Failure "Unparsed input following type".
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# `1 + F`;;
Exception:
Failure
 "typechecking error (initial type assignment): F has type bool, it cannot
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- Theorems can only be created (ultimately) by applying a small number of primitive rules