HOL Light from the foundations (part 2/3)

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HOL types

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- A polymorphic type variable

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The type system is very closely analogous to that of OCaml itself, and HOL's parser even uses similar algorithms to assign most general polymorphic types.

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# `SUC 0`;;
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- Abstractions or lambdas, written with a backslash

```
# `\x. x + 1`;;
val it : term = `\x. x + 1`
```

HOL Light primitive rules (1)

$$\frac{}{\vdash t = t}\ \texttt{REFL}$$

$$\frac{\Gamma \vdash s = t \quad \Delta \vdash t = u}{\Gamma \cup \Delta \vdash s = u}\ \texttt{TRANS}$$

$$\frac{\Gamma \vdash s = t \quad \Delta \vdash u = v}{\Gamma \cup \Delta \vdash s(u) = t(v)}\ \texttt{MK\_COMB}$$

$$\frac{\Gamma \vdash s = t}{\Gamma \vdash (\lambda x.\ s) = (\lambda x.\ t)}\ \texttt{ABS}$$

$$\frac{}{\vdash (\lambda x.\ t)x = t}\ \texttt{BETA}$$

HOL Light primitive rules (2)

$$\frac{}{\{p\} \vdash p} \ \texttt{ASSUME}$$

$$\frac{\Gamma \vdash p = q \quad \Delta \vdash p}{\Gamma \cup \Delta \vdash q} \ \texttt{EQ\_MP}$$

$$\frac{\Gamma \vdash p \quad \Delta \vdash q}{(\Gamma - \{q\}) \cup (\Delta - \{p\}) \vdash p = q} \ \texttt{DEDUCT\_ANTISYM\_RULE}$$

$$\frac{\Gamma[x_1, \ldots, x_n] \vdash p[x_1, \ldots, x_n]}{\Gamma[t_1, \ldots, t_n] \vdash p[t_1, \ldots, t_n]} \ \texttt{INST}$$

$$\frac{\Gamma[\alpha_1, \ldots, \alpha_n] \vdash p[\alpha_1, \ldots, \alpha_n]}{\Gamma[\gamma_1, \ldots, \gamma_n] \vdash p[\gamma_1, \ldots, \gamma_n]} \ \texttt{INST\_TYPE}$$

HOL's logical connectives

The usual logical connectives are given ASCII renderings:

$\bot$	`F`	Falsity
$\top$	`T`	Truth
$\neg$	`~`	Not
$\wedge$	`/\`	And
$\vee$	`\/`	Or
$\Rightarrow$	`==>`	Implies ('if ... then ...')
$\Leftrightarrow$	`<=>`	Iff ('... if and only if ...')
$\forall$	`!`	For all
$\exists$	`?`	There exists
$\exists!$	`?!`	There exists a unique

The definitions of the logical connectives

HOL Light is so foundational that even all the basic logical connectives are *defined* in terms of equality:

$$
\begin{array}{rcl}
\top & = & (\lambda p.\ p) = (\lambda p.\ p) \\
\wedge & = & \lambda p.\ \lambda q.\ (\lambda f.\ f\ p\ q) = (\lambda f.\ f\ \top\ \top) \\
\Rightarrow & = & \lambda p.\ \lambda q.\ p \wedge q = p \\
\forall & = & \lambda P.\ P = \lambda x.\ \top \\
\exists & = & \lambda P.\ \forall q.\ (\forall x.\ P(x) \Rightarrow q) \Rightarrow q \\
\vee & = & \lambda p.\ \lambda q.\ \forall r.\ (p \Rightarrow r) \Rightarrow (q \Rightarrow r) \Rightarrow r \\
\bot & = & \forall p.\ p \\
\neg & = & \lambda p.\ p \Rightarrow \bot \\
\exists! & = & \lambda P.\ \exists P \wedge \forall x.\ \forall y.\ P\ x \wedge P\ y \Rightarrow (x = y)
\end{array}
$$

The usual properties of the connectives are *derived* from the primitive rules.

Basic syntax functions

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- Destructor functions dest_var, dest_const, dest_comb and dest_abs to break down terms of various kinds

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# dest_comb `SUC 0`;;
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- Corresponding constructors `mk_var`, `mk_const`, `mk_comb` and `mk_abs`

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val it : term * term = (`SUC`, `0`)
```

- Corresponding constructors `mk_var`, `mk_const`, `mk_comb` and `mk_abs`

```
# mk_var("p",`:bool`);;
val it : term = `p`
```

- `frees` to get the free variables in a term

```
# frees `x + y + 1`;;
val it : term list = [`x`; `y`]
```

Representing more complex terms

All the expressions in logic and mathematics are ultimately expressed using just those four basic terms, and one can explore how it is done using the destructor functions

- Binary logical connectives are just curried functions of the appropriate type:

```
# dest_comb `p /\ q`;;
val it : term * term = (`(/\) p`, `q`)
```

- Quantifiers are higher-order functions applied to an abstraction

```
# dest_comb `!x. x < x + 1`;;
val it : term * term = (`(!)`, `\x. x < x + 1`)
```

Getting help

Note that one can also get help on any predefined HOL Light functions using the `help` function, e.g.

```
# help "mk_abs";;
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```
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There is also a full Reference manual with the same information.

Basic and derived definitional principles

Basic principle of constant definition

The only primitive constant for the logic itself is equality `=` with polymorphic type $\alpha \to \alpha \to \texttt{bool}$.

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$$\vdash c = t$$

This is an object-level definitional principle, in that c is a constant, not some meta-level abbreviation. It is easy to see that this is conservative, and in particular consistency-preserving.

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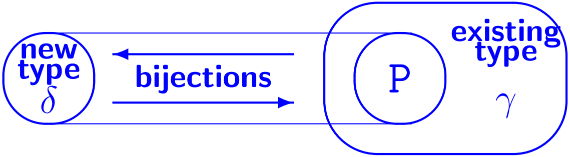
Later we add an infinite type `ind` (individuals) to assert the axiom of infinity.

All other types are introduced by `new_basic_type_definition`, the rule of type definition, to be in bijection with any nonempty subset of an existing type.



Again, this is conservative and consistency-preserving.

HOL as a definitional framework

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One of the innovations of Gordon's original HOL work was to extend this 'correct-by-construction' approach to the definitions of new concepts, which works very nicely in a general framework like HOL, so:

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Just using axioms was compared by Russell to theft in place of honest toil.

Convenient higher-level definitional principles

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HOL Light supports all these and more using safely *derived* definitional principles.

Inductively defined relations

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```
# new_inductive_definition `E(0) /\ (!n. E(n) ==> E(n + 2))`;;
val it : thm * thm * thm =
  (|- E 0 /\ (!n. E n ==> E (n + 2)),
   |- !E'. E' 0 /\ (!n. E' n ==> E' (n + 2)) ==> (!a. E a ==> E' a),
   |- !a. E a <=> a = 0 \/ (?n. a = n + 2 /\ E n))
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```

The function returns a triple of theorems:

- A 'rule' theorem (the inductively defined predicate is closed under the rules)
- An 'induction' or minimality theorem (the inductively defined predicate is the least such)
- A 'cases' theorem that each element arises by virtue of one of the rules.

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The rule returns a pair of theorem, one justifying 'structural induction' over the type:

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val btree_INDUCT : thm =
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and the other justifying definition by primitive recursion

```
val btree_RECURSION : thm =
|- !f0 f1.
        ?fn. (!a. fn (Leaf a) = f0 a) /\
             (!a0 a1. fn (Branch a0 a1) = f1 a0 a1 (fn a0) (fn a1))
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Some tail-recursive cases can be justified even without an ordering:

```
define `collatz(n) = if n <= 1 then n
                     else if EVEN(n) then collatz(n DIV 2)
                     else collatz(3 * n + 1)`;;
```