

# Formalizing Categories: Univalent Double Categories

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# Overview

- **Aim:** Formalizing double categories.
- **Motivation I:** Formalization is cool!
- **Motivation II:** Double categories have become prominent.
- **Challenge I:** Equivalences of double categories are tricky!
- **Challenge II:** Choosing appropriate coherences of double categories not completely understood.

# Why formalize categories

- ① Used in mathematics:
  - Needed for geometry, algebra, topology, ...
  - Concrete example: Liquid Tensor Experiment.
- ② Helps us understand categories:
  - In Coq UniMath we have two “notions” of categories:
    - **Categories with sets of objects:** Isomorphism invariant
    - **Univalent categories:** Equivalence invariant

# Categories in Type Theory

A category can be defined in a dependent type theory as expected:

- 1 Type of objects  $\mathcal{O} : U$
- 2 Type of morphisms  $\mathcal{M}$  dependent on  $\mathcal{O} \times \mathcal{O} \rightarrow U$
- 3 Identity:  $\prod x : \mathcal{O}, \mathcal{M}_{xx}$ .
- 4 Composition:  $\prod xyz : \mathcal{O}, \mathcal{M}_{xy} \rightarrow \mathcal{M}_{yz} \rightarrow \mathcal{M}_{xz}$ .
- 5 Unity:  $\prod (xy : \mathcal{O})(f : x \rightarrow y), (f \circ id_x = f) \times (id_y \circ f = f)$ .
- 6 Associativity:  $\prod (xyz : \mathcal{O})(f : x \rightarrow y)(g : y \rightarrow z)(h : z \rightarrow w), ((f \circ g) \circ h = f \circ (g \circ h))$ .

# Extensional vs. Intensional

We can construct “the universe of categories”  $\mathcal{C}at$  using  $\Sigma$ -types. What are the identities of this type?

- In Lean (an extensional TT) identities are a property and are given by strict equality.
- In Coq UniMath (an intensional TT) identities are structures. By adding key properties we get interesting universes.
  - $\mathcal{C}at_{Set}$ : Categories such that  $\mathcal{O}$  is a set  
 $\Rightarrow$  identities in  $\mathcal{C}at_{Set}$  are isomorphisms.
  - $\mathcal{C}at_{Univ}$ : Categories that are *univalent*<sup>1</sup>  
 $\Rightarrow$  identities in  $\mathcal{C}at_{Univ}$  are equivalences.

## Remark

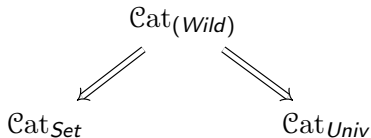
We need to assume that  $\mathcal{M}_{xy}$  is a set to get the behavior we want.

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<sup>1</sup>Benedikt Ahrens, Krzysztof Kapulkin, and Michael Shulman. Univalent categories and the Rezk completion.

# Summary

Coq UniMath enables us to study categorical notions up to the desired notion of “sameness”.



## Remark

“Categories that are both univalent and have type of objects sets”

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“Univalent categories with at most one iso between two objects”

# 2-Categories

Let's see more examples in 2-category theory. A 2-category has:

- 1 Objects
- 2 1-Morphisms between objects
- 3 Unital and associative composition of 1-morphisms
- 4 2-Morphisms between 1-morphisms
- 5 Unital and associative composition of 2-morphisms

# Classifying 2-Categories

In classical literature, 2-categories can be classified up to several layers of strictness:

- (1) Isomorphism
- (2) Essentially surjective & local isomorphism
- (2') Equivalence of underlying category & local isomorphism of 2-morphisms
- (3) Essentially surjective & local equivalence at the level of the categories formed by the 1-morphisms and 2-morphisms.



## 2-Categories in Coq UniMath

Following the philosophy we outlined above, what we would like are 3 characterizations of 2-categorical data with various levels of strictness, such that their identities correspond to the three classes of equivalences stated before:

- 1 2-categories with a set of objects and set of 1-morphisms  $2\mathcal{C}at_{set, set}$
- 2 2-categories with underlying univalent 1-category  $2\mathcal{C}at_{univ, set}$
- 3 Univalent bicategories  $bi\mathcal{C}at_{Univ}$ .<sup>2</sup>

### Remark

Similar to above, we want our 2-morphisms to always form sets!

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<sup>2</sup>Benedikt Ahrens, Dan Frumin, Marco Maggesi, Niccolò Veltri, and Niels van der Weide. Bicategories in univalent foundations. Math. Structures Comput. Sci. 2021

# Danger! Danger!

## Fact

In order to capture equivalences of 2-categories we cannot only add a univalence condition, we have to relax the whole 2-categorical structure to a bicategory (need to have non-trivial unitor and associators).

So, we get a **category theory/UniMath formalization** feedback loop:

- 1 Formalization can help study categorical structures up to desired equivalences.
- 2 This necessitates formalizing the appropriately weakened structures.
- 3 This needs categorical understanding of appropriate definition.

# Upshot

## Conclusion

Formalizing categorical structures via Coq UniMath provides a framework for studying the structure precisely up to the desired level of sameness.

This needs correctly choosing definitions that are appropriately weak!

# Finding Univalent Double Categories

The aim of our work is to develop appropriate univalence conditions for double categories! Double categories have found applications in various places

- Have been used to define limits of 2-categories.<sup>3</sup>
- Relevant in applied category theory (Lenses, ...) <sup>4</sup>
- Various applications in computer science<sup>5,6</sup>

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<sup>3</sup>tsllil clingman and Lyne Moser. Bi-initial objects and bi-representations are not so different. *Cah. Topol. Géom. Différ. Catég.*, 63(3):259–330, 2022

<sup>4</sup>Bryce Clarke. The double category of lenses. 2023.

<sup>5</sup>John C. Baez, Kenny Courser, and Christina Vasilakopoulou. Structured versus decorated cospans. *Compositionality*, 4(3):39, 2022.

<sup>6</sup>Pierre-Évariste Dagand and Conor McBride. A categorical treatment of ornaments. (LICS 2013)

# What are double categories?

- **Idea:** A category with two types of morphisms.
- **Precise Notion:** A category object in the category of categories.
- **Explicit precise notion:**
  - Objects
  - Horizontal morphisms depending on objects
  - Vertical morphisms depending on objects
  - Squares depending on horizontal/vertical morphisms
  - Identities for objects, horizontal and vertical morphisms
  - Compositions of horizontal, vertical morphisms, squares
  - Unitality of compositions
  - Associativity of compositions

# Equivalences of double categories

Double categories come with various notions of equivalences:

- 1 Isomorphisms
- 2 Horizontal equivalences: Equivalences on the category of objects and horizontal morphisms & the category of vertical morphisms and squares.
- 3 Vertical equivalences: The other way around
- 4 Gregarious equivalences: Tricky, but symmetric.

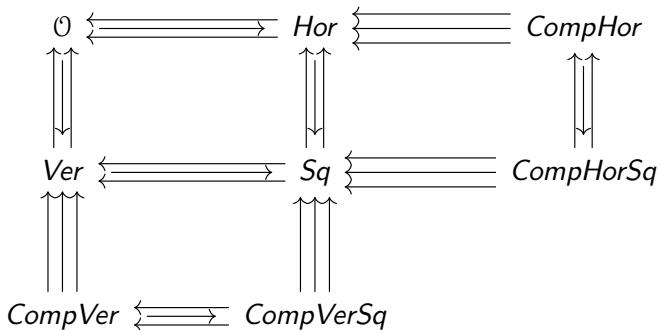
**Want:** Notions of suitably weak double categories fitting all these types of equivalences that we can formalize!

## Remark

There is some necessary math to make the formalization work!

# Double categorical framework

Let's start with the basic input in type theory for double categories, which we capture via the following diagram:



We are missing unitality and associativity, and needed strictness.

# Defining double categories for isomorphisms

This one is as straightforward as the ones before:

## Proposition

*In double categories in which:*

- *The types of objects, the two morphisms and squares are sets*
  - *All associativity and unitality conditions are given strictly*
- the identities correspond to isomorphisms.*



# Defining double categories for vertical univalence

For the next one, it gets hard!

- Objects and vertical morphisms should form a univalent category.
- Horizontal morphisms and squares should be a univalent category.
- Objects and horizontal morphisms can neither be univalent nor have a set of objects... so composition has to be non-strict in general. Here we need associators and unitors.

⇒ We have to weaken the structure in one direction! Such definition exists under the name “pseudo-double category” by Grandis<sup>7</sup>, Johnson–Yau<sup>8</sup>, ... .

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<sup>7</sup>Marco Grandis. Higher dimensional categories. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020. From double to multiple categories.

<sup>8</sup>Niles Johnson and Donald Yau. 2-dimensional categories. Oxford University Press, Oxford, 2021.

# Univalent double categories for vertical equivalences.

Having all this input the claim now is that things work as one might expect that we have formalized using Coq UniMath.

**Proposition (Ahrens–North–R.–van der Weide)**

*Identities in the universe of pseudo-double categories correspond to vertical equivalences.*

Of course we have similar results if we use horizontal equivalences.

# Technical sidenote

We use the method of displayed categories to actually formalize things!

- Displayed categories were introduced as a technical method to add structures, such as monoidal structures to categories.
- The approach is modular allowing for effective formalizations.
- As part of this work we generalized displayed categories to 2-sided displayed categories, which we use to formalize pseudo double categories and establish the desired univalence.

# Gregarious Univalence

The next step is to develop an appropriately weak notion of double category for gregarious equivalences. Unfortunately we know very little:

- 1 The literature on weaker notions of double categories is very sparse. There is a notion of “double bicategory” by Verity.<sup>9</sup>
- 2 The literature on gregarious equivalences is also sparse. It was conjectured that they provide a good theory of equivalences (aka model structure), but not proven.

This points to the following conjecture we are aiming towards:

## Conjecture

Identities of Verity double bicategories (or a slight variation thereof) correspond to gregarious equivalences.

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<sup>9</sup> Dominic Verity. Enriched categories, internal categories and change of base. Repr. Theory Appl. Categ., (20):1–266, 2011.

# Challenges towards gregarious Univalence

We face two major challenges:

- 1 On the mathematical side, the sparse literature necessitates developing some of the results first.
- 2 On the formalization side, we cannot use 2-sided displayed categories anymore, and need more sophisticated libraries.

# The End

Thank you!

- Formalization:

<https://github.com/UniMath/UniMath/tree/master/UniMath/Bicategories/DoubleCategories>

- Questions:

- 1 Ask me in person!
- 2 Email: [rasekh@mpim-bonn.mpg.de](mailto:rasekh@mpim-bonn.mpg.de)