PolySAT
A Word-level Solver for Large Bitvectors

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Background: Satisfiability Modulo Theories (SMT)

Problem Statement:

Is φ satisfiable?

where φ is a formula in classical first-order logic with equality and certain theories (e.g., fragments of integer arithmetic).

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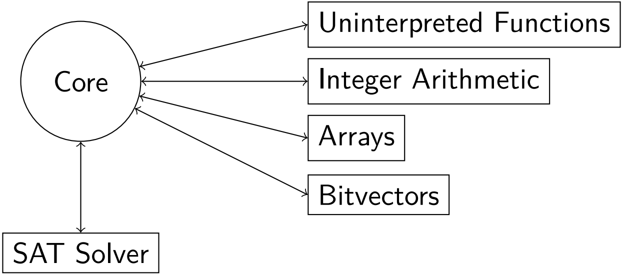
$$f(x+1) \neq f(x) \wedge 2x + 5y \leq 10 \wedge (x = y \vee f(x) = f(y))$$

Background: SMT Solving

SMT Solver: fully automated system to determine satisfiability



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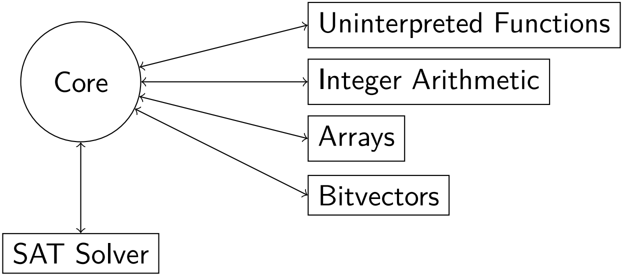

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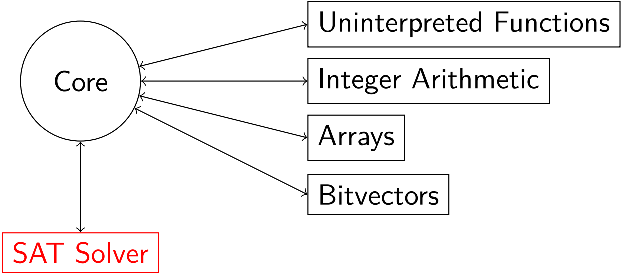

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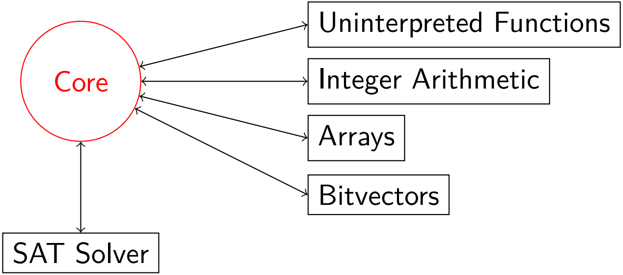

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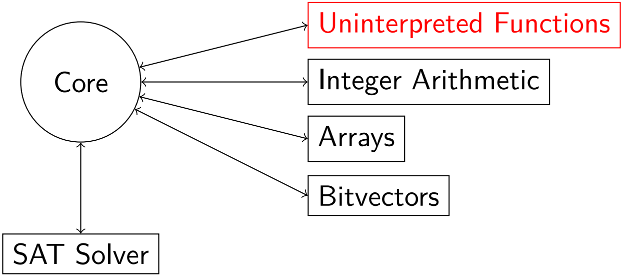

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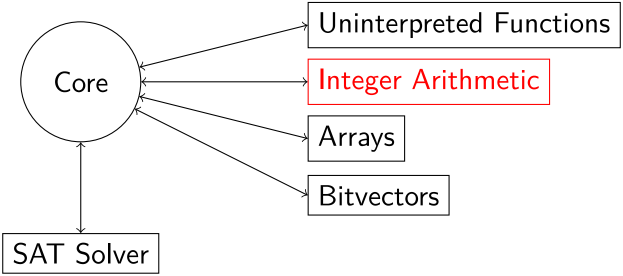

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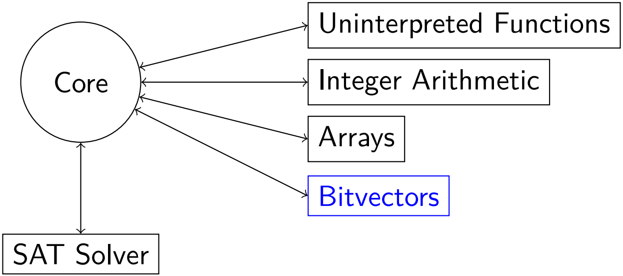

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Our focus: theory solver for bitvectors!

PolySAT: a Word-level Solver for Large Bitvectors

Bitvectors?

1. Sequence of bits, e.g., `01011`
2. Fixed-width machine integers, e.g., `uint32_t`, `int64_t`
3. Modular arithmetic: $\mathbb{Z}/2^k\mathbb{Z}$

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Examples:

- $2x^2y + z = 3$
- $x + 3 \leq x + y$
- $z = x \,\&\, y$
- $x[3{:}0] = 0$

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Natural target for many program verification tasks!

Bitvector Pitfalls

$\mathbb{Z}/2^k\mathbb{Z}$ is a finite commutative ring, but not a field.

Ordering: representatives $\{0, 1, \ldots, 2^k - 1\}$ (unsigned bitvectors)

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$x, y \geq 0 \;\not\Rightarrow\; xy \geq x$	Overflow/wraparound: $3 \cdot 6 = 2 \mod 2^4$
$x, y \neq 0 \;\not\Rightarrow\; xy \neq 0$	Zero divisors: $6 \cdot 8 = 0 \mod 2^4$
$x \leq y \;\not\Rightarrow\; x - y \leq 0$	Usual inequality normalization fails

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Example

$x + 3 \leq x + y \mod 2^3$

- For $x = 0$: $3 \leq y \iff y \in \{3, 4, 5, 6, 7\}$
- For $x = 2$: $5 \leq 2 + y \iff y \in \{3, 4, 5\}$

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- For $x = 0$: $3 \leq y \iff y \in \{$
- For $x = 2$: $5 \leq 2 + y \iff y \in \{$
- $x + 3 \leq -y + 2 \mod 2^3$

$$
\begin{aligned}
&p \leq q \\
&p \leq p - q - 1 \\
&q - p \leq q \\
&q - p \leq -p - 1 \\
&-q - 1 \leq -p - 1 \\
&-q - 1 \leq p - q - 1
\end{aligned}
$$

Solving Approaches

- Bit-blasting

  Translate into boolean formula and use SAT solver

[1] Yoni Zohar et al.: *Bit-Precise Reasoning via Int-Blasting*

[2] S. Graham-Lengrand, D. Jovanović, B. Dutertre: *Solving bitvectors with MCSAT: explanations from bits and pieces*

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- MCSAT-based approaches[2]

  Search for assignment to bitvector variables

  $\rightsquigarrow$ PolySAT

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PolySAT Overview

- Theory solver for bitvector arithmetic
  - Input: conjunction of bitvector constraints
  - Output: SAT or UNSAT
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- Theory solver for bitvector arithmetic
  - Input: conjunction of bitvector constraints
  - Output: SAT or UNSAT
- Based on modular integer arithmetic ($\mathbb{Z}/2^k\mathbb{Z}$)
- Search for a model of the input constraints
  - Incrementally assign bitvector variables
  - Keep track of viable values for variables
  - Add lemmas on demand

Bitvector Language

Arithmetic	$x + y$, $x \cdot y$, div, mod, ...
Equations	$x = y$
Inequalities	$x \leq y$ with $x, y \in \{0, 1, \ldots, 2^k - 1\}$
Inequalities (signed)	$x \leq_s y$ with $x, y \in \{-2^{k-1}, \ldots, 2^{k-1} - 1\}$
Bit-wise	and, or, xor, not, ...
Structural	$shift$, $concat$, $extract$, ...

Bitvector Constraints in PolySAT

Inequalities	$p \leq q$	(polynomials p, q)
Overflow	$\Omega^*(p, q)$	
Bit-wise	$r = p \,\&\, q$	
Structural	$r = p \ll q, \quad r = p \gg q, \quad x = y[h{:}l]$	
Clauses	Disjunction of constraint literals	

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Quotient/remainder	$q := \mathtt{bvudiv}(a, b)$, $r := \mathtt{bvurem}(a, b)$	

- $a = bq + r$
- $\neg\Omega^*(b, q)$
- $\neg\Omega^+(bq, r)$ (e.g., $bq \leq -r - 1$)
- $b \neq 0 \rightarrow r < b$
- $b = 0 \rightarrow q + 1 = 0$

PolySAT Solving Loop

Modified CDCL loop, similar to MCSAT[3]

- Assign boolean values to constraint literals ($p \leq q$ vs. $p > q$)
- Assign integer values to bitvector variables ($x \mapsto 3$)

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- Assign boolean values to constraint literals ($p \leq q$ vs. $p > q$)
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Main components:

- Trail Γ records assignments and reasons
- For each variable x, keep track of viable values V_x
- Conflict $\mathcal{C}$: set of constraints that contradicts Γ
- Conflict analysis extracts lemmas from $\mathcal{C}$

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Example: Polynomial Equations

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C_1: \quad & x^2y + 3y + 7 = 0 \mod 2^4 \\
C_2: \quad & 2y + z + 8 = 0 \mod 2^4 \\
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   Conflict: $\mathcal{C} = \{C_3, x = 0, y = 3, z = 2\}$

Example: Polynomial Equations (conflict)

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Follow dependencies of $\mathcal{C}$ according to Γ:

$$\mathcal{C}' = \{C_3, x = 0, y = 3, C_2\}$$

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Lemma:

$$C_3 \wedge C_2 \rightarrow 3x + 1 = 0$$

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   Conflict due to parity!

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   Conflict due to parity!
7. Unsatisfiable.

How to choose values?

For each variable x, keep track of viable values V_x:

- choose a value from V_x for decisions
- propagate $x \mapsto v$ when $V_x = \{v\}$ is a singleton set
- conflict if $V_x = \emptyset$

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Currently:

- V_x represented as set of intervals
- when x appears only linearly, extract a forbidden interval
- additionally, keep track of fixed bits of x (e.g., $2^4 x = 2^4\, 5$)
- bit-blasting as fallback
  (only a single bitvector variable)

Intervals

We use half-open intervals:

- Usual notation $[\ell; u[$
- but wrap around if $\ell > u$

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Examples mod 2^4:

$$[2; 5[= \{2, 3, 4\}$$
$$[13; 2[= \{13, 14, 15, 0, 1\}$$
$$[0; 0[= \emptyset$$

Note:

$$p \in [\ell; u[\quad \iff \quad p - \ell < u - \ell$$

Forbidden Intervals

p, q, r, s: polynomials, evaluable in current trail Γ
x: variable, unassigned in Γ

$$px + r \leq qx + s$$

[4] S. Graham-Lengrand, D. Jovanović, B. Dutertre: *Solving bitvectors with MCSAT: explanations from bits and pieces*

Forbidden Intervals

p, q, r, s: polynomials, evaluable in current trail Γ
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$$px + r \leq qx + s$$

$\widehat{p}$	$\widehat{q}$	Interval	
0	1	$x \notin [-s; r - s[$	if $r \neq 0$
1	0	$x \notin [s - r + 1; -r[$	if $s \neq -1$
1	1	$x \notin [-s; -r[$	if $r \neq s$
		Lemmas from intervals[4]	

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		Lemmas from intervals[4]	
$\{0, n\}$	$\{0, n\}$	Set of intervals ("equal coeff.")	
n	m	Set of intervals ("disequal coeff.")	
		Intervals from fixed bits	
		Combination with value selection	
		Fallback to bit-blasting	

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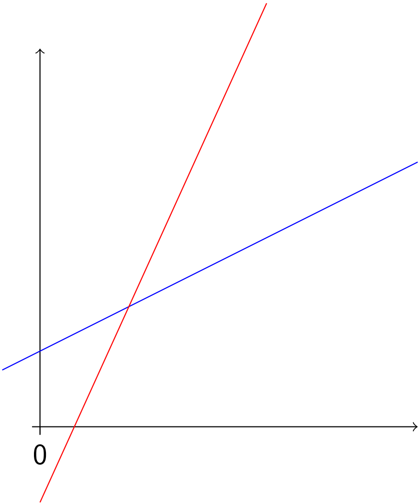
Forbidden Intervals (disequal coefficients)

$$px + r \leq qx + s \quad \text{with } \widehat{p} \neq \widehat{q}$$



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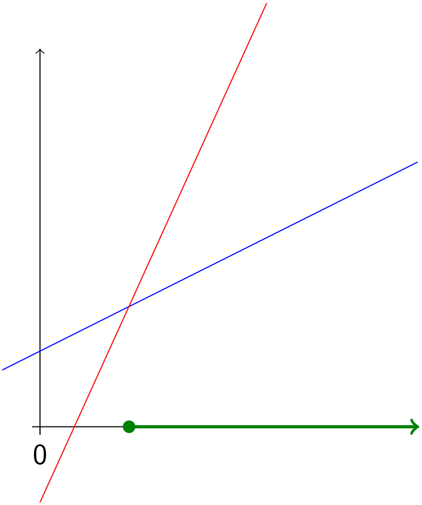

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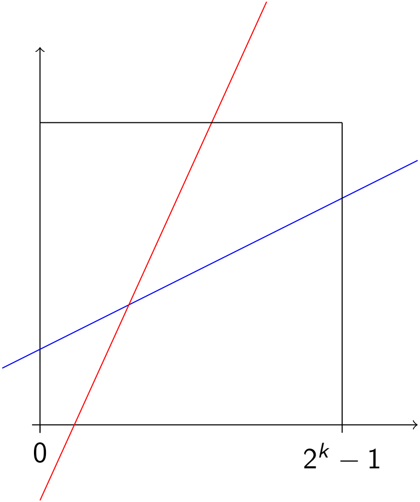
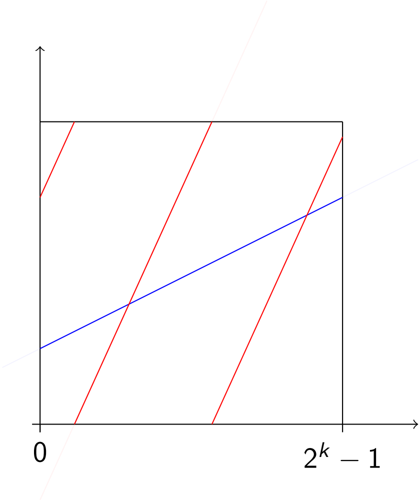


Forbidden Intervals (disequal coefficients)

$$\color{blue}{px + r} \leq \color{red}{qx + s} \quad \text{with } \widehat{p} \neq \widehat{q}$$



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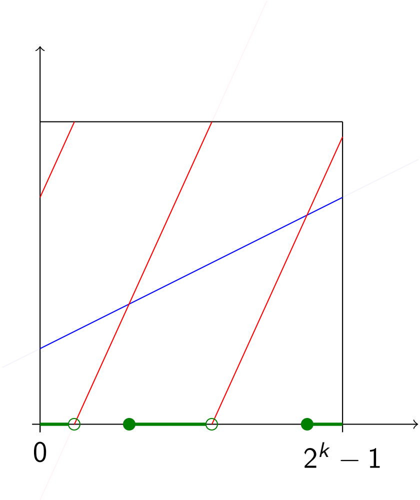

choice based on current value $\widehat{x}$

Conflict Resolution Strategy

1. Track the conflict's cone of influence while backtracking over the trail Γ
2. Conflict resolution plugins derive lemmas from constraints in the conflict
3. For now, accumulate lemmas from conflict plugins
   - New (often simpler) constraints improve propagation
   - Easy to experiment with new types of lemmas
4. When reaching the first relevant decision, learn lemmas and resume search

Forbidden Interval Lemma

- Assume conflict $V_x = \emptyset$
- Forbidden intervals:
  $C_i \implies x \notin [\ell_i; u_i[$ if c_i

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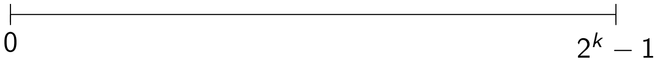

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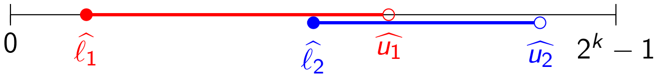

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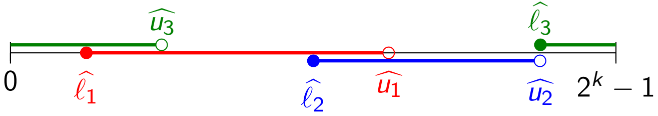

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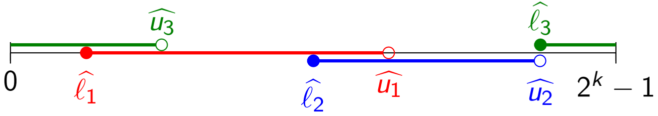

- Use symbolic intervals to express the overlap condition:

$$u_1 \in [\ell_2; u_2[\ \wedge \ u_2 \in [\ell_3; u_3[\ \wedge \ u_3 \in [\ell_1; u_1[$$

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Conflict Resolution Plugins

Forbidden Intervals Lemma			
Superposition	$p(x) = 0 \wedge q(x) = 0$	$\Longrightarrow$	$rp(x) + sq(x) = 0$
	choose r, s to eliminate highest power of x		
Overflow	$\Omega^*(p, q) \wedge \neg\Omega^*(p, r)$	$\Longrightarrow$	$q > r$
Inequality	$px < qx$	$\Longrightarrow$	$\Omega^*(x, p) \vee p < q$
	$px \leq qx$	$\Longrightarrow$	$\Omega^*(x, p) \vee p \leq q$
			$\vee\ x = 0$
Bit-wise *and*	$x = p \,\&\, q$	$\Longrightarrow$	$x \leq p$
	$x = p \,\&\, q \wedge p = q$	$\Longrightarrow$	$x = p$
	$x = p \,\&\, q \wedge p = 2^n - 1$	$\Longrightarrow$	$2^{n-k}x = 2^{n-k}q$
Bounds	$C(x, y) \wedge x \in [x_l; x_h]$	$\Longrightarrow$	$y \in [y_l; y_h]$
	$\Omega^*(p, q) \wedge p \leq b_1$	$\Longrightarrow$	$q \geq b_2$
	$axy + bx + cy + d \leq \dots$	$\Longrightarrow$	$\dots$
...	...		...

Conclusion

PolySAT

- Bit-vector solver in Z3
- Word-level reasoning

Thank you!