Amenable Groups in Lean

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Here, $g \cdot f \in \ell^\infty(G, \mathbb{R})$ is defined by

$$(g \cdot f)(x) := f(g^{-1} \cdot x).$$

Examples

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$$m(f) := \frac{1}{|G|} \cdot \sum_{x \in G} f(x).$$

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- F_2 is *not* amenable.

Why amenable groups?

Theorem (Banach-Tarski-Paradox)

There exists a disjoint partition

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- $\mathbb{Z}$ is amenable.

DEMO

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Proof.

Define $m : \ell^\infty(\mathbb{Z}, \mathbb{R}) \to \mathbb{R}$ by

$$m(f) := \frac{1}{|\mathbb{Z}|} \sum_{x \in \mathbb{Z}} f(x).$$

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$\mathbb{Z}$ is amenable.

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Define $m : \ell^\infty(\mathbb{Z}, \mathbb{R}) \to \mathbb{R}$ by

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$$m(f) := \lim_{n \in \omega} \frac{1}{|F_n|} \sum_{x \in F_n} f(x).$$

where ω is a non-principal ultrafilter on $\mathbb{N}$.

- By the ultrafilter convergence theorem, the limit exists.
- Check: This defines a left-invariant mean.

Remark

Works for a Følner sequence $(F_n)_{n \in \mathbb{N}}$ (e.g. $F_n := \{-n, \dots, n\}$).

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Example

$F_n := \{-n, \ldots, n\}$ for $G = \mathbb{Z}$.

DEMO

Conclusions

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- doable, and also fun

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- doable, and also fun
- better understanding

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- doable, and also fun
- better understanding
- well-suited for some applications

THANKS!

Check out the code on my webpage:
`https://homepages.uni-regensburg.de/~usm34387/`

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