Pen-and-paper type theory:

Understanding a modal type theory via a categorical model

Florrie Verity
Australian National University

Interactions of proof assistants and mathematics
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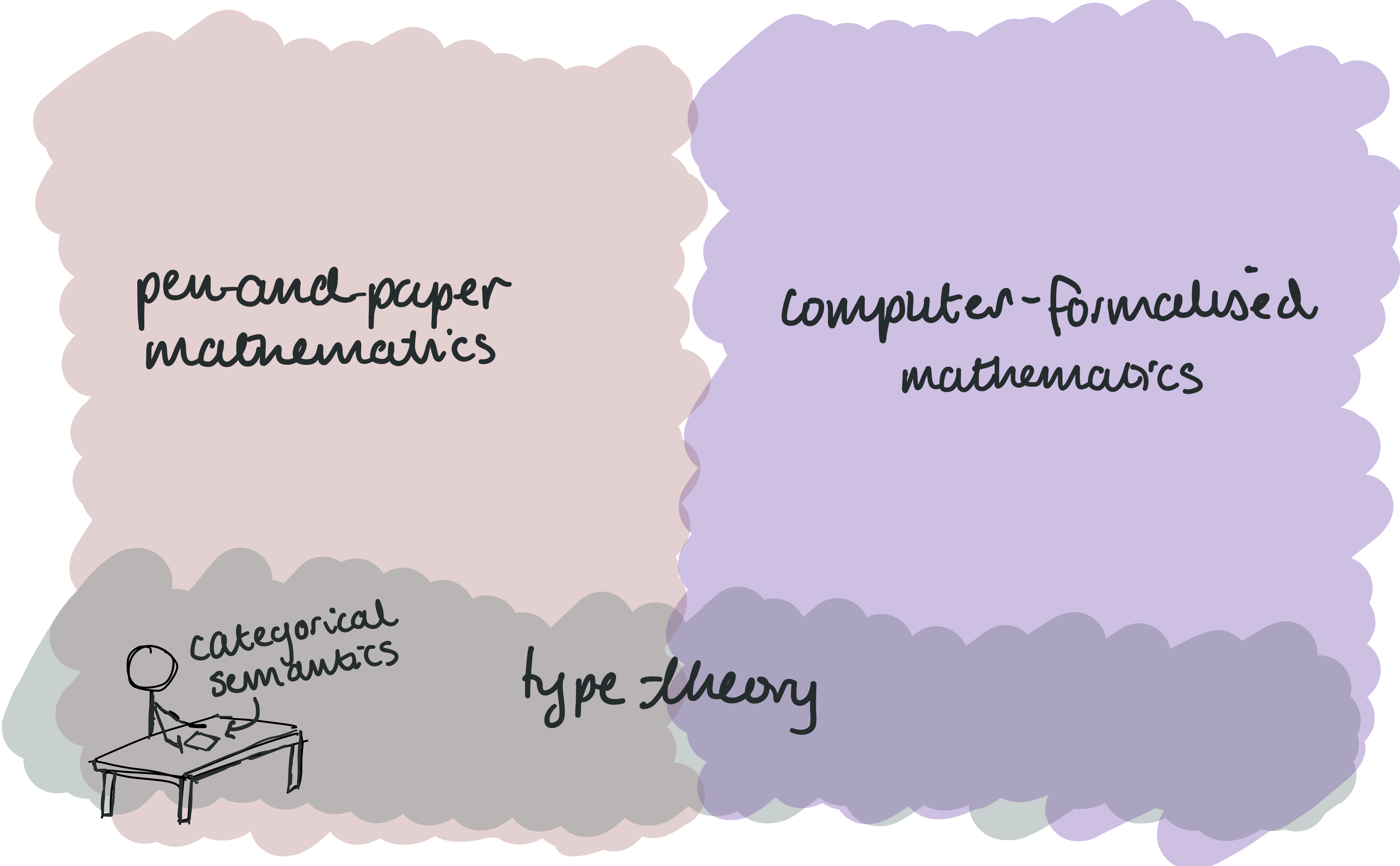
The view from where I'm sitting



Choose your own adventure

You encounter a type theory you haven't seen before!
Do you -

A) try to implement it using your favourite proof assistant? *

B) try to understand its categorical semantics?

key:
* = interaction with a proof assistant

The scenario

- Want to understand "crisp type theory" - a modal type theory

$A \rightsquigarrow$ "A is true"
$\Box A \rightsquigarrow$ "A is necessarily true"
$\Diamond A \rightsquigarrow$ "A is possibly true"

- Why?
  - important role in cubical models of HoTT

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* crisp type theory has been implemented in Agda
  - "Agda-Flat" (Vezzosi)

* myriad interactions of HoTT with proof assistants

Motivation

- HoTT has models in "presheaf categories"
  - simplicial sets (Voevodsky)
  - cubical sets (Coquand, Orton & Pitts, Awodey)

- Two descriptions in a presheaf category $\hat{\mathbb{C}}$:

  1. Category-theoretic via diagrams in $\hat{\mathbb{C}}$

     (Awodey, Gambino & Sattler, ...)

  2. Logical via the "internal type theory" of $\hat{\mathbb{C}}$

     (Coquand et al, Orton & Pitts, ...)

     * Some of this is formalised in Agda

Motivation

Example: a "trivial fibration structure" on

① ... p is a choice of diagonal fillers $j(m,u,v)$

$$\begin{tikzcd} S \arrow[r, "u"] \arrow[d, "m"'] & A \arrow[d, "p"] \\ T \arrow[r, "v"'] \arrow[ur] & X \end{tikzcd}$$

for all $m \in Cof$ such that

$$\begin{tikzcd} t^*(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[r, "u"] \arrow[d] & A \arrow[d, "p"] \\ T' \arrow[r, "t"'] \arrow[urr] & T \arrow[r, "v"'] \arrow[ur] & X \end{tikzcd}$$

for all $m \in Cof$, for all $t : T' \to T$.

② ... $\alpha : X \to U$ is an element

$$t : TFib(\alpha)$$

where

$$TFib(\alpha) = \prod_{\varphi : \Phi} \prod_{v : \alpha^{\{\varphi\}}} \sum_{a : \alpha} v = \lambda(a)$$

? How do you relate ① and ②?

Motivation

- use the technique of "Kripke-Joyal forcing", generalised from propositions to types
  (Awodey, Gambino & Hazratpour, 2021)

⇝ precise relation of the descriptions

Problem

the "universe of uniform fibrations"

①

$$\begin{tikzcd} \mathrm{Fib}^*(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fill}(\alpha \circ x)_I \arrow[d] \\ X \arrow[r, "\eta"'] & (X^I)_I \end{tikzcd}$$

② impossible!

Solution

- extend internal type theory with the modal operator of crisp type theory. (Licata, Orton, Pitts & Spitters, 2018)

Crisp type theory

- a fragment of Shulman's "spatial type theory", part of "real-cohesive HoTT" (2018)
- dependent version of Pfenning and Davies' modal type theory (2001)
- Features "flat" modality $\flat A$ $\Box A$
- Features "split contexts"
  - standard context – $x_1 : \alpha_1, x_2 : \alpha_2, \dots, x_n : \alpha_n$
  - split context – $\underbrace{x_1 : \delta_1, \dots, x_n : \delta_n \mid y_1 : \gamma_1, \dots, y_m : \gamma_m}_{\Delta \mid \Gamma}$

crisp variables | standard variables

$x_1 : \flat\delta_1, \dots, x_n : \flat\delta_n, y_1 : \gamma_1, \dots, y_m : \gamma_m$

Crisp type theory

- Crisp types depend only on crisp variables

$$\frac{\Delta \mid \cdot \vdash \alpha \text{ type}}{\Delta \mid \Gamma \vdash \flat\alpha \text{ type}}$$

- Two kinds of context extension

① standard context extension

$$\frac{\Delta \mid \Gamma \vdash \alpha \text{ type}}{\Delta \mid \Gamma, x : \alpha \vdash}$$

② extension of the crisp context

$$\frac{\Delta \mid \cdot \vdash \alpha \text{ type}}{\Delta, x : \alpha \mid \cdot \vdash}$$

Modelling dependent type theory

Let $\mathbb{C}$ be a category, $\mathcal{D}$ be a class of maps in $\mathbb{C}$.

Ingredients of a type theory:

- contexts Γ $\quad\leftrightsquigarrow\quad$ objects Γ in $\mathbb{C}$
- types $\Gamma \vdash \alpha \text{ type}$ $\quad\leftrightsquigarrow\quad$ arrows $\begin{tikzcd} \Gamma.\alpha \arrow[d] \\ \Gamma \end{tikzcd}$ in $\mathcal{D}$
- terms $\Gamma \vdash a : \alpha$ $\quad\leftrightsquigarrow\quad$ sections $\begin{tikzcd} \Gamma.\alpha \arrow[d] \\ \Gamma \arrow[u, bend left, "a"] \end{tikzcd}$

The problem of substitution

- substitution of a term into a type

$$\frac{x:\alpha \vdash \beta(x)\ \text{type} \qquad y:\gamma \vdash t:\alpha}{y:\gamma \vdash \beta(t)\ \text{type}} \quad \leftrightsquigarrow$$

pullback

$$\begin{tikzcd} \beta(t) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \beta \arrow[d] \\ \gamma \arrow[r, "t"'] & \alpha \end{tikzcd}$$

substitution is strictly functorial, so

$$\frac{x:\alpha \vdash \beta(x)\ \text{type} \qquad y:\gamma \vdash t:\alpha \qquad z:\delta \vdash r:\gamma}{z:\delta \vdash \beta(t)(r) = \beta(t \circ r)\ \text{type}}$$

but pullback is only pseudofunctorial,

$$\beta(t)(r) \cong \beta(t \circ r)$$

One solution: Natural models (Awodey 2016, Fiore)

A natural model is a category $\mathbb{C}$ with data

i) a terminal object 1

ii) a "universe" (locally representable natural transformation)
$ty : \tilde{\mathcal{U}} \longrightarrow \mathcal{U}$ in $\widehat{\mathbb{C}}$.

Recall ingredients of a type theory:

- contexts Γ $\quad\leftrightsquigarrow\quad$ objects Γ in $\mathbb{C}$

  empty context $\bullet$ $\quad\leftrightsquigarrow\quad$ terminal object 1 in $\mathbb{C}$

- types $\Gamma \vdash \alpha$ type $\quad\not\leftrightsquigarrow\quad$ arrows $\begin{array}{c}\Gamma.\alpha\\ \downarrow\\ \Gamma\end{array}$ in D

  $\leftrightsquigarrow$ arrows $\begin{array}{c}\Gamma.\alpha\\ p\downarrow\\ \Gamma\end{array}$ in $\mathbb{C}$ with

$$\begin{tikzcd} \text{よ}(\Gamma.\alpha) \arrow[r] \arrow[d, "\text{よ}p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{U}} \arrow[d] \\ \text{よ}(\Gamma) \arrow[r, "\alpha"'] & \mathcal{U} \end{tikzcd} \quad \text{in } \widehat{\mathbb{C}}$$

Modelling crisp type theory

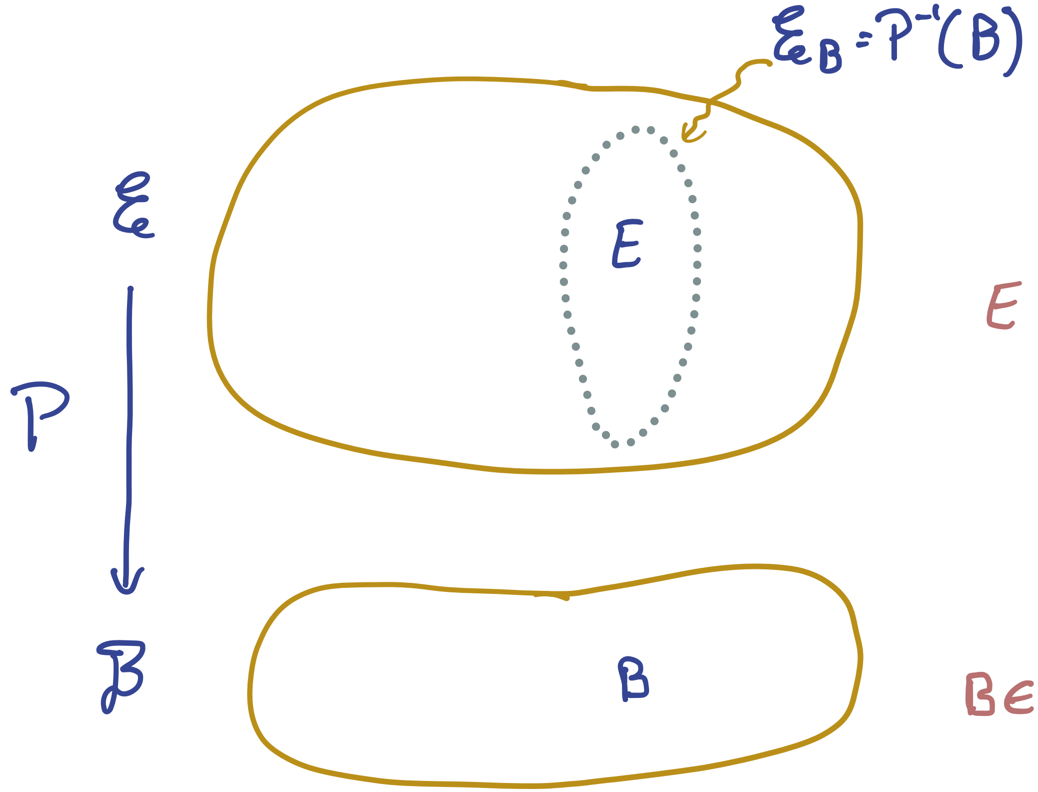
- Conjectured model in Licata et. al. (2018), from Shulman (2018)

$$\begin{tikzcd} \text{constant presheaf on } B & \widehat{\mathbb{C}} \arrow[loop right, "\flat"] \arrow[d, bend left, "p_*"] & X \arrow[d, mapsto] \\ B \arrow[u, mapsto] & \mathrm{Set} \arrow[u, bend left, "p^*"] \arrow[u, phantom, "\dashv"] & \{1 \xrightarrow{x} X\} \text{ global elements} \end{tikzcd}$$

- Our strategy - zoom out

Modelling crisp type theory

For a context $\Delta \mid \Gamma$, want to capture the dependency of Γ on Δ.



$E \in \mathcal{E}_B \leftrightsquigarrow \Delta \mid \Gamma$

$B \in \mathcal{B} \leftrightsquigarrow \Delta$

Modelling crisp type theory

Idea Equip

(i) the base category, and
(ii) each fibre

with the structure to model a type theory.

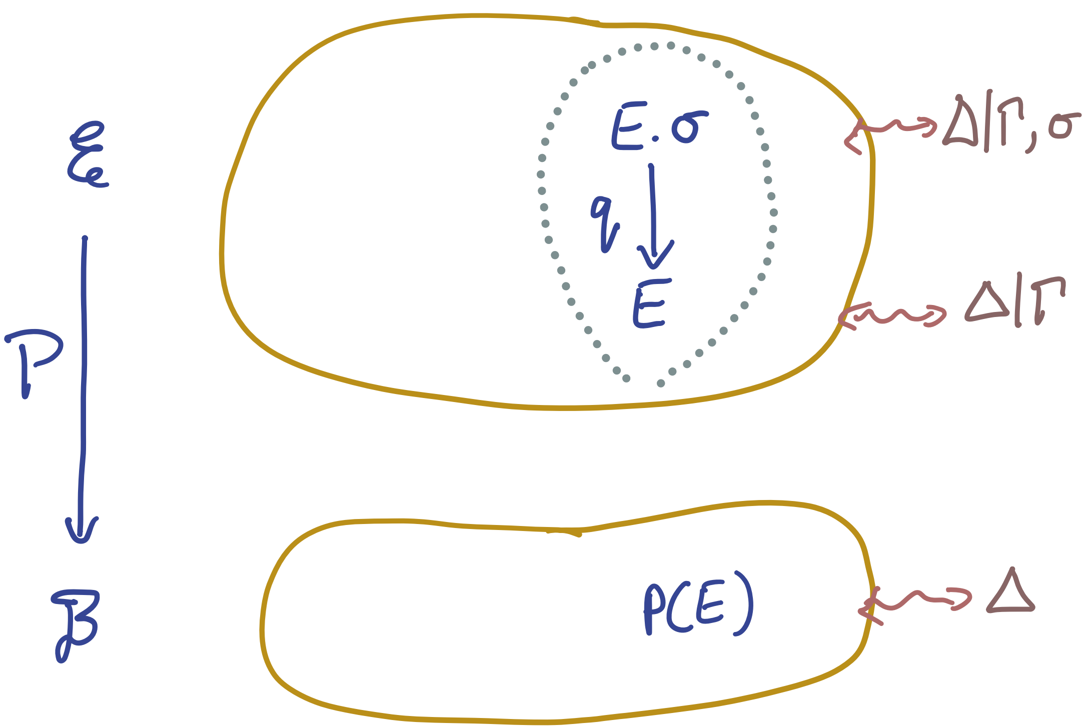
Recall This structure for a natural model is

(i) a terminal object
(ii) a universe

(ii) Universes - fibrewise in $\hat{\mathcal{E}}$

Regular context extension:

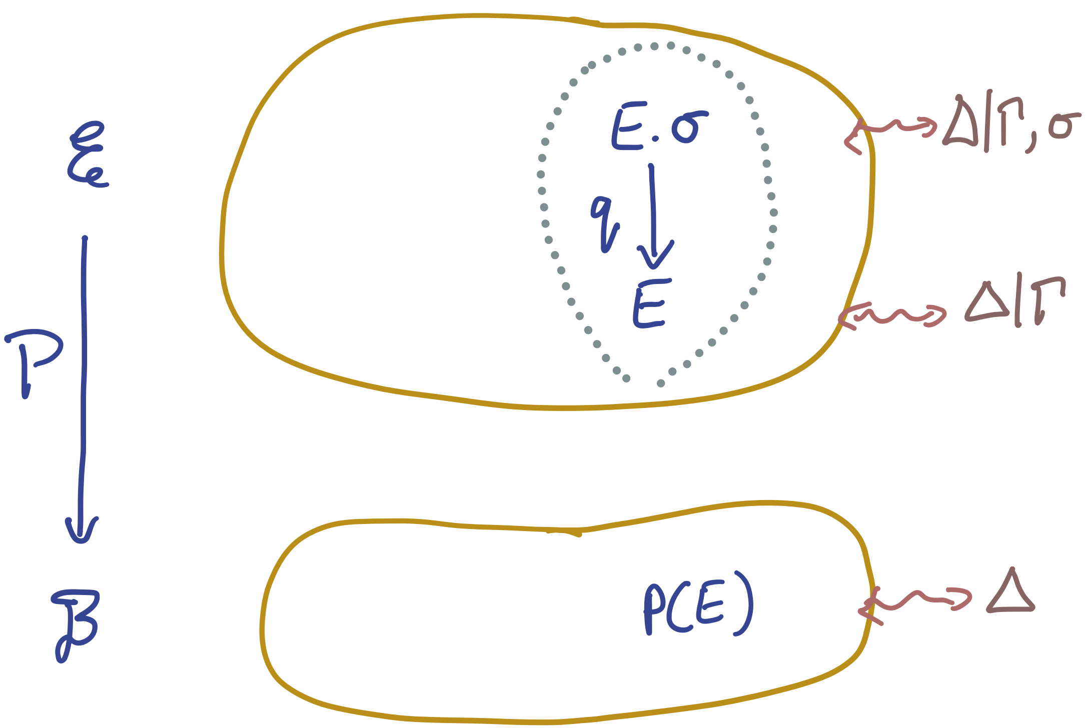
$$\frac{\Delta \mid \Gamma \vdash \sigma \text{ type}}{\Delta \mid \Gamma, \sigma \vdash}$$



(ii) Universes – fibrewise in $\widehat{\mathcal{E}}$

Regular context extension:

$$\frac{\Delta \mid \Gamma \vdash \sigma \text{ type}}{\Delta \mid \Gamma, \sigma \vdash}$$



To implement:

Ask for a universe in $\widehat{\mathcal{E}}$ s.t. in the specified pullback along $\sigma : \text{よ}E \to \mathcal{U}_{\mathcal{E}}$,

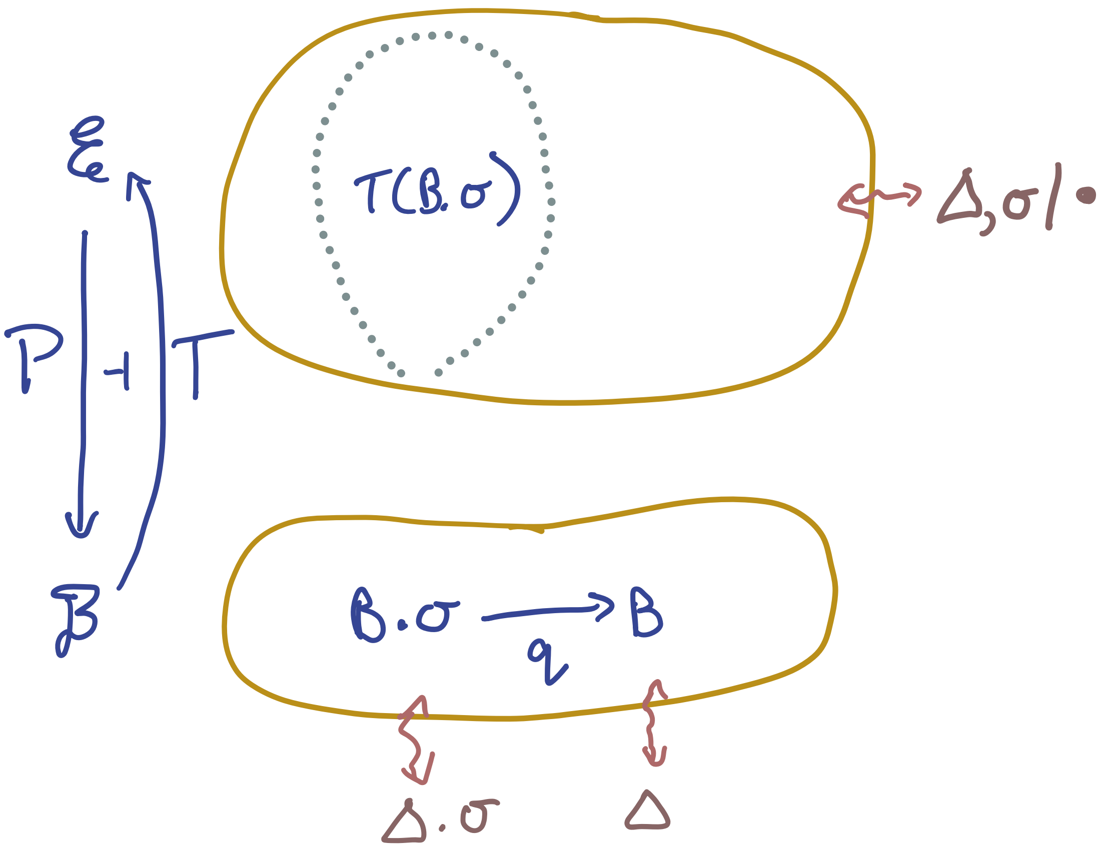
$$\begin{tikzcd}
\text{よ}E.\sigma \arrow[r] \arrow[d, "\text{よ}q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathcal{U}}_{\mathcal{E}} \arrow[d, "ty"] \\
\text{よ}E \arrow[r, "\sigma"'] & \mathcal{U}_{\mathcal{E}}
\end{tikzcd},$$

$E.\sigma \xrightarrow{p} E$ in $\mathcal{E}$ lies in the fibre $\mathcal{E}_{P(E)}$.

(ii) Universe - in $\hat{\mathcal{B}}$

Crisp context extension:

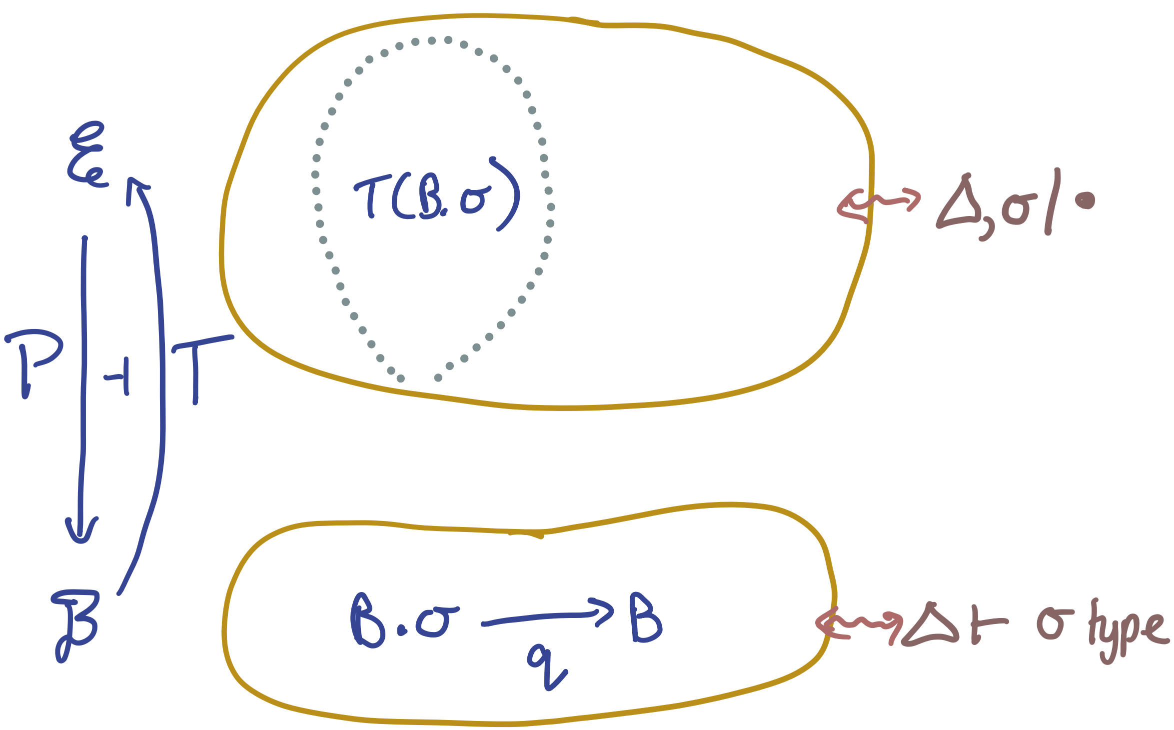
$$\frac{\Delta \mid \bullet \vdash \sigma \ \mathsf{type}}{\Delta, \sigma \mid \bullet \vdash}$$



(ii) Universe - in $\widehat{\mathcal{B}}$

Crisp context extension:

$$\frac{\Delta \mid \bullet \vdash \sigma \ \mathsf{type}}{\Delta, \sigma \mid \bullet \vdash}$$



To implement:

Ask that the map in $\widehat{\mathcal{B}}$ defined

$$\begin{tikzcd}
\tilde{\mathcal{U}}_{\mathcal{B}} : \mathcal{B}^{op} \arrow[r, "T^{op}"] \arrow[d] & \mathcal{E}^{op} \arrow[r, "\tilde{\mathcal{U}}_{\mathcal{E}}"] & \mathsf{Set} \\
\mathcal{U}_{\mathcal{B}} : \mathcal{B}^{op} \arrow[r, "T^{op}"] & \mathcal{E}^{op} \arrow[r, "\mathcal{U}_{\mathcal{E}}"] & \mathsf{Set}
\end{tikzcd}$$

is a universe.

So we have:

$$\begin{tikzcd}
よ B.\sigma \arrow[r] \arrow[d, "よ q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{U}}_{\mathcal{E}} \circ T^{op} \arrow[d, "ty \circ T^{op}"] \\
よ B \arrow[r, "\sigma"'] & \mathcal{U}_{\mathcal{E}} \circ T^{op}
\end{tikzcd}$$

The universe for $\mathcal{B}$ is defined relative to the universe for $\mathcal{E}$

$\Rightarrow$ there is a correspondance between typing judgements

$$\Delta \vdash_{\mathcal{B}} \sigma \text{ type}$$

and $\Delta | \bullet \vdash_{\mathcal{E}} \sigma \text{ type}$

in $\widehat{\mathcal{B}}$, $\quad \text{よ}_B \xrightarrow{\ \sigma\ } \mathcal{U}_{\mathcal{E}} \circ T^{op} \quad \longleftrightarrow \quad \Delta \vdash_{\mathcal{B}} \sigma \text{ type}$

$$\overline{\overline{\qquad\qquad\qquad}} \text{ yoneda}$$

$$\sigma \in \mathcal{U}_{\mathcal{E}}(T(B))$$

$$\overline{\overline{\qquad\qquad\qquad}} \text{ yoneda}$$

in $\widehat{\mathcal{E}}$, $\quad \text{よ}_{T(B)} \xrightarrow{\ \sigma\ } \mathcal{U}_{\mathcal{E}} \quad \longleftrightarrow \quad \Delta | \bullet \vdash_{\mathcal{E}} \sigma \text{ type}$

The abstract model

Let $P : \mathcal{E} \longrightarrow \mathcal{B}$ be a functor.

Axioms

1) P has a right adjoint right inverse, T.
2) $\mathcal{B}$ has a specified terminal object.
3) There is a locally representable map

$$ty : \tilde{\mathcal{U}}_{\mathcal{E}} \longrightarrow \mathcal{U}_{\mathcal{E}} \text{ in } \widehat{\mathcal{E}}$$

whose local representatives are given fibrewise.

4) $\tilde{\mathcal{U}}_{\mathcal{E}} \circ T^{op} \longrightarrow \mathcal{U}_{\mathcal{E}} \circ T^{op}$ in $\widehat{\mathcal{B}}$ is locally representable.

(+ ask for cartesian lifts of display maps in $\mathcal{B}$)

Claim: This models the context in crisp type theory.

Zooming back in

The intended model

constant presheaf on B

$$\begin{tikzcd}
\text{constant presheaf on } B & \hat{\mathbb{C}} \arrow[d, bend left, "p_*"] \arrow[d, phantom, "\dashv"] & X \arrow[d, mapsto] \\
B \arrow[u, mapsto] & \mathrm{Set} \arrow[u, bend left, "p^*"] & \{1 \xrightarrow{x} X\}
\end{tikzcd}$$

b

global elements

satisfies the axioms of our abstract model, where

$$\begin{tikzcd}
\mathcal{E} := \hat{\mathbb{C}} \downarrow \mathrm{Set} \arrow[d, "\mathrm{cod}"'] \\
\mathcal{B} := \mathrm{Set}
\end{tikzcd}$$

Conclusions

- "Relativised, fibrewise" natural model structure
- The abstract model gives a picture of two interacting type theories that's proving useful to work with
    - returned to Kripke-Joyal forcing work
- the model remains to be formalised as semantics
    - perhaps a task to do in a proof assistant

Thanks